

GLOBAL WELL-POSEDNESS FOR THE TWO DIMENSIONAL COMPRESSIBLE MHD EQUATIONS WITH LARGE DATA

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ABSTRACT. In this paper we are concerned with the global well-posedness for the compressible MHD equations with large data. We show that if the shear viscosity μ is a positive constant and the bulk viscosity λ is the power function of the density, that is, $\lambda(\rho) = \rho^\beta$ with $\beta > 3$, then the two dimensional compressible MHD system with the periodic boundary conditions on the torus \mathbb{T}^2 have a unique global classical solution (ρ, u, H) . In this work we extended the results about compressible Navier-Stokes equations in [42] to compressible MHD equations by applying several new techniques to overcome the coupling between velocity and magnetic field.

Key words compressible MHD equations; isentropic fluids; global well-posedness; density-dependent viscosity

1. INTRODUCTION

In this paper, we consider the following compressible Magnetohydrodynamics (MHD) equations on \mathbb{T}^2 ,

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \nabla \times H \times H + \mu \Delta u + \nabla((\mu + \lambda(\rho)) \operatorname{div} u), \\ \partial_t H - \nabla \times (u \times H) - \nu \Delta H = 0, \\ \operatorname{div} H = 0, \\ (\rho(x, t), H(x, t), u(x, t))|_{t=0} = (\rho_0(x), H_0(x), u_0(x)), \end{cases} \quad (1.1)$$

which describes the motion of electrically conducting media in the presence of a magnetic field. Here ρ , u , H and p denote the density, velocity, magnetic field and pressure respectively. $\rho_0(x)$, $u_0(x)$ and $H_0(x)$ are initial values. The pressure term $p(\rho)$ is assumed to obey the polytropic γ -law, i.e.

$$p(\rho) = a\rho^\gamma, \quad (1.2)$$

where a is the entropy constant and normalized to be one without loss of generality and $\gamma > 1$ is called the adiabatic index. Also, we assume that the functions $\mu(\rho)$, and $\lambda(\rho)$ are defined on $[0, +\infty)$ and satisfy the conditions

$$\mu(\rho) = \text{const} > 0, \quad \lambda(\rho) = \rho^\beta, \quad \beta > 3. \quad (1.3)$$

And \mathbb{T}^2 is the 2 dimensional torus $[0, 1] \times [0, 1]$ and $t \in [0, T]$ for any fixed $T > 0$.

For smooth initial data such that the density ρ_0 is bounded and bounded away from zero (i. e. $0 < \underline{\rho} \leq \rho_0(x) \leq M$), existence and uniqueness of local classical solutions to the compressible Navier-Stokes equations have been known for a long

time (see the pioneering work of J. Nash [35] or the paper of N. Itaya [25]). Matsumura and Nishida [33] proved the global well-posedness for compressible Navier-Stokes equations for smooth data close to equilibrium. The reader may refer to [12, 15, 16, 17, 18, 19, 20, 24, 29] for more recent advances on the subject. Particularly, Danchin has obtained several important well-posedness in critical spaces for compressible Navier-Stokes equations [15, 16, 17, 18, 19]. Chen-Miao-Zhang [12] have proved the local well-posedness in $\dot{B}_{2,1}^1 \times (\dot{B}_{2,1}^0)^2$ for the viscous shallow water equations and for compressible Navier-Stokes equations with density dependent viscosities in the Besov spaces $\dot{B}_{p,1}^{\frac{N}{p}}$ [11]. Bian and Yuan have obtained local well-posedness in the critical Besov spaces [1] and super critical Besov spaces [2] for the compressible MHD equations. Concerning the global existence of weak solutions to compressible Navier-Stokes equations for the large initial data, readers refer to [3, 4, 30, 34], and refer to [5, 12, 43, 22] and references therein for the viscous shallow water equations. Recently, Vaigant-Kazhikhov [42] obtained global well-posedness of strong and classical solution for the compressible Navier-Stokes system with large data and without vacuum. Jiu-Wang-Zhou [26] generalized this result to the case which may contain vacuums with the periodic boundary conditions on the torus \mathbb{T}^2 .

Due to the physical importance and mathematical challenges, the study on (1.1) has attracted many physicists and mathematicians [21, 31, 32]. Existence and uniqueness of (weak, strong or smooth) solutions in one dimension can be found in [9, 10, 23, 27, 28] and the references cited therein. Construction of global classical or strong solutions to the Navier-Stokes equations in the high-dimensional case was open, and more difficult for the system (1.1) since the velocity and magnetic couple and there are more nonlinear terms. This paper is devoted to construct global classical solution for the 2-dimensional compressible MHD equations (1.1) with large data. we extended the results in [42] by applying several new techniques to overcome the coupling between velocity and magnetic field.

In the absence of heat conduction, it was proved by Z. P. Xin that any non-zero smooth solution with initially compact supported density would blow up in finite time (see [44]). This result was generalized to the cases for the non-barotropic compressible Navier-Stokes system with heat conduction [13] and for non-compact but rapidly decreasing at far field initial densities [38]. As a reasonable starting point, we will therefore restrict our work to solutions such that ρ remains positive.

Our result is expressed in the following.

Theorem 1.1. *If the initial data (ρ_0, u_0, H_0) satisfy that*

$$\begin{aligned} 0 < (\rho_0, p(\rho_0)) &\in W^{2,q}(\mathbb{T}^2) \times W^{2,q}(\mathbb{T}^2), \\ (u_0, H_0) &\in H^2(\mathbb{T}^2) \times H^2(\mathbb{T}^2), \quad \int_{\mathbb{T}^2} \rho_0 dx > 0. \end{aligned} \tag{1.4}$$

Then there exists a unique classical global solution to the 2D MHD system (1.1)–(1.3) and satisfies

$$\begin{aligned} 0 < \rho \leq C, \quad (\rho, p(\rho)) &\in C([0, T]; W^{2,q}(\mathbb{T}^2)), \quad \rho_t \in C([0, T]; L^q(\mathbb{T}^2)), \\ (u, H) &\in C([0, T]; H^2(\mathbb{T}^2)) \cap L^2([0, T]; H^3(\mathbb{T}^2)), \\ (u_t, H_t) &\in L^2([0, T]; H^1(\mathbb{T}^2)), \quad (\sqrt{\rho}u_t, H_t) \in L^\infty([0, T]; L^2(\mathbb{T}^2)). \end{aligned} \tag{1.5}$$

Remark 1.2. We did not make any effort to optimize the assumptions on the initial data or the function space that is used in the analysis. The whole point is to construct global in time classical solutions.

Notation: Throughout the paper, C stands for a "harmless" constant, which is independent of $m, t \in [0, T]$. We sometimes use the notation C_α for some generic constant depending only on α . we use the notation $A \lesssim B$ as an equivalent of $A \leq CB$. The notation f_+ means that $f_+ = \max\{0, f\}$. The notation $L^p(\mathbb{T}^2)$, $1 \leq p \leq \infty$, stands for the usual Lebesgue spaces on \mathbb{T}^2 and $\|\cdot\|_p$ denotes its L^p norm, and without ambiguity, we write $\int f(x)dx$ instead of $\int_{\mathbb{T}^2} f(x)dx$.

2. PRELIMINARIES

First, we state some assertions that are used later.

Lemma 2.1. *For every function $u \in W_0^{1,m}(\mathbb{T}^2)$ or $u \in W^{1,m}(\mathbb{T}^2)$ with $\int_{\mathbb{T}^2} u dx = 0$, it holds that*

$$\|u\|_q \leq C \|\nabla u\|_m^\theta \|u\|_r^{1-\theta},$$

where $\theta = (\frac{1}{r} - \frac{1}{q})(\frac{1}{r} - \frac{1}{m} + \frac{1}{2})^{-1}$, and if $m < 2$, then q is between r and $\frac{2m}{2-m}$, that is, $q \in [r, \frac{2m}{2-m}]$ if $r < \frac{2m}{2-m}$, $q \in [\frac{2m}{2-m}, r]$ if $r \geq \frac{2m}{2-m}$, if $m = 2$, then $q \in [r, +\infty)$, if $m > 2$, then $q \in [r, +\infty]$. Consequently, for every function $u \in W^{1,m}(\mathbb{T}^2)$, one has

$$\|u\|_q \leq C(\|u\|_1 + \|\nabla u\|_m^\theta \|u\|_r^{1-\theta}),$$

where C is a constant which may depend on q .

The above Lemma is the Gagliardo-Nirenberg inequality which can be found in [36, 42]. The following Lemma is the Poincare inequality.

Lemma 2.2. [45, 37, 41] *For every function $u \in W_0^{1,m}(\mathbb{T}^2)$ or $u \in W^{1,m}(\mathbb{T}^2)$ with $\int_{\mathbb{T}^2} u dx = 0$, if $1 \leq m < 2$, then*

$$\|u\|_{\frac{2m}{2-m}} \leq C(2-m)^{-\frac{1}{2}} \|\nabla u\|_m,$$

where the positive constant C is independent of m .

From Lemma 2.2, we can prove the following Lemma, of which proof can also be found in [42].

Lemma 2.3. *For every function $u \in W^{1, \frac{2m}{m+\eta}}(\mathbb{T}^2)$ with $m \geq 2$ and $0 < \eta \leq 1$, we have*

$$\|u\|_{2m} \leq C(\|u\|_1 + m^{\frac{1}{2}} \|u\|_{2(1-\varepsilon)}^s \|\nabla u\|_{\frac{2m}{m+\eta}}^{1-s}),$$

where $\varepsilon \in [0, \frac{1}{2}]$, $s = \frac{(1-\varepsilon)(1-\eta)}{m-\eta(1-\varepsilon)}$ and the positive constant C is independent of m .

By virtue of the maximum principle, we can prove the following lemma. For brevity, we state it here without proof.

Lemma 2.4. *If the initial datum $|H_0| \leq C$, and the function H satisfies*

$$H_t + u \cdot \nabla H - \nu \Delta H - H \cdot \nabla u + H \operatorname{div} u = 0,$$

then $|H| \leq C$, with C a positive constant.

Second, we introduce the following notations:

$$F = (2\mu + \lambda(\rho))\operatorname{div} u - p(\rho) - \frac{1}{2}|H|^2, \quad w = \partial_{x_1} u_2 - \partial_{x_2} u_1, \quad (2.1)$$

$$B = \frac{1}{\rho}(\mu w_{x_1} + (F + \frac{1}{2}|H|^2)_{x_2}), \quad L = \frac{1}{\rho}(-\mu w_{x_2} + (F + \frac{1}{2}|H|^2)_{x_1}). \quad (2.2)$$

Using them and recalling that $\mu = \text{const}$, from system (1.1), we get

$$\begin{aligned} u_{1t} + u \cdot \nabla u_1 - \frac{1}{\rho} H \cdot \nabla H_1 &= \frac{1}{\rho}(-\mu \omega_{x_2} + F_{x_1}) = L - \frac{1}{2\rho}|H|_{x_1}^2, \\ u_{2t} + u \cdot \nabla u_2 - \frac{1}{\rho} H \cdot \nabla H_2 &= \frac{1}{\rho}(\mu \omega_{x_1} + F_{x_2}) = B - \frac{1}{2\rho}|H|_{x_2}^2. \end{aligned}$$

Then ω and F solve the following system:

$$\begin{cases} \omega_t + u \cdot \nabla \omega + \omega \operatorname{div} u + (\frac{1}{\rho} H \cdot \nabla H_1)_{x_2} - (\frac{1}{\rho} H \cdot \nabla H_1)_{x_1} \\ = (B - \frac{1}{2\rho}|H|_{x_2}^2)_{x_1} - (L - \frac{1}{2\rho}|H|_{x_1}^2)_{x_2}, \\ (F + \frac{1}{2}|H|^2)_t + u \cdot \nabla (F + \frac{1}{2}|H|^2) + (2\mu + \lambda(\rho))[(u_{1x_1})^2 + 2u_{1x_2}u_{2x_1} \\ + (u_{2x_2})^2] - \rho(2\mu + \lambda(\rho))[(F + \frac{1}{2}|H|^2)(\frac{1}{2\mu + \lambda(\rho)})' + (\frac{p}{2\mu + \lambda(\rho)})'] \operatorname{div} u \\ - \frac{2\mu + \lambda(\rho)}{\rho}[(H_{1x_1})^2 + 2H_{1x_2}H_{2x_1} + (H_{2x_2})^2] - (2\mu + \lambda(\rho)) \\ [(\frac{1}{\rho})_{x_1} H \cdot \nabla H_1 + (\frac{1}{\rho})_{x_2} H \cdot \nabla H_2] \\ = (2\mu + \lambda(\rho))[(B - \frac{1}{2\rho}|H|_{x_2}^2)_{x_2} + (L - \frac{1}{2\rho}|H|_{x_1}^2)_{x_1}]. \end{cases} \quad (2.3)$$

Using the form of the functions $B(x, t)$ and $L(x, t)$ and the continuity equation, together with the magnetic equation, the system for (B, L, H) can be derived as

$$\begin{cases} \rho B_t + \rho u \cdot \nabla B - \rho B \operatorname{div} u + u_{x_2} \cdot \nabla (F + \frac{1}{2}|H|^2) + \mu u_{x_1} \nabla \omega + \mu[\omega \operatorname{div} u \\ + (\frac{1}{\rho} H \cdot \nabla H_1)_{x_2} - (\frac{1}{\rho} H \cdot \nabla H_1)_{x_1}]_{x_1} + \{(2\mu + \lambda(\rho))[(u_{1x_1})^2 + 2u_{1x_2}u_{2x_1} \\ + (u_{2x_2})^2]\}_{x_2} - \{\frac{2\mu + \lambda(\rho)}{\rho}[(H_{1x_1})^2 + 2H_{1x_2}H_{2x_1} + (H_{2x_2})^2]\}_{x_2} \\ - \{\rho(2\mu + \lambda(\rho))[(F + \frac{1}{2}|H|^2)(\frac{1}{2\mu + \lambda(\rho)})' + (\frac{p}{2\mu + \lambda(\rho)})'] \operatorname{div} u\}_{x_2} \\ - \{(2\mu + \lambda(\rho))[(\frac{1}{\rho})_{x_1} H \cdot \nabla H_1 + (\frac{1}{\rho})_{x_2} H \cdot \nabla H_2]\}_{x_2} \\ = \mu\{(B - \frac{1}{2\rho}|H|_{x_2}^2)_{x_1} - (L - \frac{1}{2\rho}|H|_{x_1}^2)_{x_2}\}_{x_1} + \{(2\mu + \lambda(\rho)) \\ [(B - \frac{1}{2\rho}|H|_{x_2}^2)_{x_2} + (L - \frac{1}{2\rho}|H|_{x_1}^2)_{x_1}]\}_{x_2}, \\ \rho L_t + \rho u \cdot \nabla L - \rho L \operatorname{div} u + u_{x_1} \cdot \nabla (F + \frac{1}{2}|H|^2) - \mu u_{x_2} \nabla \omega - \mu[\omega \operatorname{div} u \\ + (\frac{1}{\rho} H \cdot \nabla H_1)_{x_2} - (\frac{1}{\rho} H \cdot \nabla H_1)_{x_1}]_{x_2} + \{(2\mu + \lambda(\rho))[(u_{1x_1})^2 + 2u_{1x_2}u_{2x_1} \\ + (u_{2x_2})^2]\}_{x_1} - \{\frac{2\mu + \lambda(\rho)}{\rho}[(H_{1x_1})^2 + 2H_{1x_2}H_{2x_1} + (H_{2x_2})^2]\}_{x_1} \\ - \{\rho(2\mu + \lambda(\rho))[(F + \frac{1}{2}|H|^2)(\frac{1}{2\mu + \lambda(\rho)})' + (\frac{p}{2\mu + \lambda(\rho)})'] \operatorname{div} u\}_{x_1} \\ - \{(2\mu + \lambda(\rho))[(\frac{1}{\rho})_{x_1} H \cdot \nabla H_1 + (\frac{1}{\rho})_{x_2} H \cdot \nabla H_2]\}_{x_1} \\ = -\mu\{(B - \frac{1}{2\rho}|H|_{x_2}^2)_{x_1} - (L - \frac{1}{2\rho}|H|_{x_1}^2)_{x_2}\}_{x_2} + \{(2\mu + \lambda(\rho)) \\ [(B - \frac{1}{2\rho}|H|_{x_2}^2)_{x_2} + (L - \frac{1}{2\rho}|H|_{x_1}^2)_{x_1}]\}_{x_1}, \\ H_t + u \cdot \nabla H - \nu \Delta H - H \cdot \nabla u + H \operatorname{div} u = 0. \end{cases} \quad (2.4)$$

These equations are equivalent to each other for the smooth solutions to the original system (1.1). In the following, we will use the above system in different steps.

3. LOWER ORDER ESTIMATES

In this section, we derive some uniform a-priori estimates. Using the method in [39, 40], we can prove the existence and uniqueness for classical solutions on a sufficiently small time interval. In what follows, we will study the global problem is connected with obtaining a priori estimates with constants only on the data of the problem and duration T of the time interval and independent of the interval of existence of a local solution. Then we can extend this solution globally. We divide the proof of the low order estimates into several steps.

Step 1. Elementary energy estimates

Lemma 3.1. *There exists a positive constant C depending on (ρ_0, u_0, H_0) , such that*

$$\begin{aligned} & \sup_{t \in [0, T]} (\|\sqrt{\rho}u\|_2^2 + \|\rho\|_\gamma^\gamma + \|\rho\|_1 + \|H\|_2^2) + \int_0^T (\|\omega\|_2^2 + \|\nabla u\|_2^2 + \|\nabla H\|_2^2 \\ & + \|(2\mu + \lambda(\rho))^{\frac{1}{2}} \operatorname{div} u\|_2^2) dt \leq C. \end{aligned} \quad (3.1)$$

Proof. Multiplying the momentum equation and the magnetic equation by u and H respectively, integrating over \mathbb{T}^2 , and then summing the resulting equations, from the continuity equation it holds that

$$\begin{aligned} & \frac{d}{dt} \int (\rho|u|^2 + |H|^2) dx + \int (\mu\omega^2 + (2\mu + \lambda(\rho))(\operatorname{div} u)^2 + \nu\|\nabla H\|_2^2) dx \\ & + \int u \cdot \nabla p dx = 0, \end{aligned} \quad (3.2)$$

where we have used the fact that

$$\Delta u = \nabla \operatorname{div} u - \nabla \times \omega,$$

and

$$u \cdot (\nabla \times H \times H) + H \cdot (\nabla \times (u \times H)) = \operatorname{div}((u \times H) \times H).$$

Note that

$$\int u \cdot \nabla p dx = \frac{d}{dt} \int \frac{\rho^\gamma}{\gamma - 1} dx. \quad (3.3)$$

Integrating the continuity equation, we have

$$\frac{d}{dt} \int \rho dx = 0, \quad (3.4)$$

which together with (3.2) and (3.3), gives that

$$\frac{d}{dt} \int (\rho|u|^2 + |H|^2 + \rho + \frac{\rho^\gamma}{\gamma - 1}) dx + \int (\mu\omega^2 + (2\mu + \lambda(\rho))(\operatorname{div} u)^2 + \nu\|\nabla H\|_2^2) dx = 0,$$

Integrating the above equality in the time variable t over $[0, T]$, and using the fact $\|\nabla u\|_2 \leq C(\|\omega\|_2 + \|\operatorname{div} u\|_2)$, it follows that

$$\begin{aligned} & \sup_{t \in [0, T]} (\|\sqrt{\rho}u\|_2^2 + \|\rho\|_\gamma^\gamma + \|\rho\|_1 + \|H\|_2^2) + \int_0^T (\|\omega\|_2^2 + \|\nabla u\|_2^2 + \|\nabla H\|_2^2 \\ & + \|(2\mu + \lambda(\rho))^{\frac{1}{2}} \operatorname{div} u\|_2^2) dt \leq C. \end{aligned}$$

□

Step 2. Density estimate

Applying the operator div to the momentum equation of (1.1), we have

$$[\text{div}(\rho u)]_t + \text{div}[\text{div}(\rho u \otimes u - H \otimes H)] = \Delta F. \quad (3.5)$$

Consider the following two elliptic problems:

$$\Delta \xi = \text{div}(\rho u), \quad \int \xi dx = 0, \quad (3.6)$$

$$\Delta \eta = \text{div}[\text{div}(\rho u \otimes u - H \otimes H)], \quad \int \eta dx = 0, \quad (3.7)$$

both with the periodic boundary condition on the torus \mathbb{T}^2 .

From the elliptic estimates and Hölder inequality, it can be derived as that

Lemma 3.2. (1) $\|\nabla \xi\|_{2m} \leq Cm \|\rho\|_{\frac{2mk}{k-1}} \|u\|_{2mk}$, for any $k > 1$, $m \geq 1$;

(2) $\|\nabla \xi\|_{2-r} \leq C \|\rho\|_{\frac{2}{2-r}} \|\sqrt{\rho} u\|_2$, for any $0 < r < 1$;

(3) $\|\eta\|_{2m} \leq Cm \|\rho\|_{\frac{2mk}{k-1}} \|u\|_{4mk}^2 + Cm \|H\|_{4m}^2$, for any $k > 1$, $m \geq 1$;

where C is a positive constant independent of m , k and r .

Proof. By the elliptic estimates to the equation (3.7) and using the Hölder inequality, we have for any $k > 1$, $m \geq 1$,

$$\|\eta\|_{2m} \leq Cm \|\rho u^2\|_{2m} + Cm \|H^2\|_{2m} \leq Cm \|\rho\|_{\frac{2mk}{k-1}} \|u\|_{4mk}^2 + Cm \|H\|_{4m}^2.$$

the statements (2) and (3) can be proved in the same way as (1). \square

Denote

$$\phi(t) = \int (\mu \omega^2 + (2\mu + \lambda(\rho))(\text{div} u)^2 + \nu \|\nabla H\|_2^2) dx, \quad (3.8)$$

and recall Lemmas 2.1-2.3 and Lemma 3.2, the following lemma holds.

Lemma 3.3. (1) $\|\xi\|_{2m} \leq Cm^{\frac{1}{2}} \|\nabla \xi\|_{\frac{2m}{m+1}} \leq Cm^{\frac{1}{2}} \|\rho\|_{\frac{1}{m}}^{\frac{1}{2}}$, for any $m \geq 2$;

(2) $\|u\|_{2m} \leq C[m^{\frac{1}{2}} \phi(t)^{\frac{1}{2}} + 1]$, for any $m \geq 2$;

(3) $\|\nabla \xi\|_{2m} \leq C[m^{\frac{3}{2}} k^{\frac{1}{2}} \|\rho\|_{\frac{2mk}{k-1}} \phi(t)^{\frac{1}{2}} + m \|\rho\|_{\frac{2mk}{k-1}}]$, for any $k > 1$, $m \geq 1$;

(4) $\|\eta\|_{2m} \leq C[m^2 k \|\rho\|_{\frac{2mk}{k-1}} \phi(t) + m \|\rho\|_{\frac{2mk}{k-1}} + m^2 \phi(t) + m]$, for any $k > 1$, $m \geq 1$;

where C is a positive constant independent of m and k .

Proof. (1) From Lemma 2.2, Lemma 3.1 and Lemma 3.2(2), clearly

$$\|\xi\|_{2m} \leq Cm^{\frac{1}{2}} \|\nabla \xi\|_{\frac{2m}{m+1}} \leq Cm^{\frac{1}{2}} \|\rho\|_{\frac{1}{m}}^{\frac{1}{2}} \|\sqrt{\rho} u\|_2 \leq Cm^{\frac{1}{2}} \|\rho\|_{\frac{1}{m}}^{\frac{1}{2}}.$$

(2) By Lemma 2.3, we get

$$\|u\|_{2m} \leq C(\|u\|_1 + m^{\frac{1}{2}} \|\nabla u\|_{\frac{2m}{m+1}}). \quad (3.9)$$

Denote $\bar{u} = \|u\|_1 = \int u dx$, then

$$|\int \rho(u - \bar{u}) dx| \leq \|\rho\|_{\gamma} \|u - \bar{u}\|_{\frac{\gamma}{\gamma-1}} \leq C \|\nabla u\|_2. \quad (3.10)$$

On the other hand, from the conservative form of the compressible MHD equations (1.1) and the periodic boundary conditions, we obtain

$$\frac{d}{dt} \int \rho(x, t) dx = \frac{d}{dt} \int \rho u(x, t) dx = 0,$$

that is,

$$\int \rho(x, t) dx = \int \rho_0(x) dx, \quad \int \rho u(x, t) dx = \int \rho_0 u_0(x) dx,$$

for any $t \in [0, T]$. Thus,

$$|\int \rho(u - \bar{u}) dx| = |\int \rho_0 u_0 dx - \bar{u} \int \rho_0(x) dx| \geq |\bar{u}| \left| \int \rho_0(x) dx - \int \rho_0 u_0(x) dx \right|,$$

which together with (3.10) implies that

$$|\bar{u}| \leq \frac{\int \rho_0 u_0(x) dx}{\int \rho_0(x) dx} + \frac{C \|\nabla u\|_2}{\int \rho_0(x) dx}.$$

Substituting the above inequality into (3.9) completes the proof of Lemma 3.3(2).

(3) By Lemma (3.2)(1) and Lemma 3.3(2), we can show that

$$\begin{aligned} \|\nabla \xi\|_{2m} &\leq Cm \|\rho\|_{\frac{2mk}{k-1}} \|u\|_{2mk} \leq Cm \|\rho\|_{\frac{2mk}{k-1}} (m^{\frac{1}{2}} k^{\frac{1}{2}} \|\nabla u\|_2 + 1) \\ &\leq C[m^{\frac{3}{2}} k^{\frac{1}{2}} \|\rho\|_{\frac{2mk}{k-1}} \phi(t)^{\frac{1}{2}} + m \|\rho\|_{\frac{2mk}{k-1}}]. \end{aligned}$$

(4) From Lemma (3.2)(3) and Lemma 3.3(2), we have

$$\begin{aligned} \|\eta\|_{2m} &\leq Cm \|\rho\|_{\frac{2mk}{k-1}} \|u\|_{4mk}^2 + Cm \|H\|_{4m}^2 \\ &\leq Cm \|\rho\|_{\frac{2mk}{k-1}} (mk \|\nabla u\|_2^2 + 1) + Cm^2 \|\nabla H\|_2^2 + Cm \\ &\leq C[m^2 k \|\rho\|_{\frac{2mk}{k-1}} \phi(t) + m \|\rho\|_{\frac{2mk}{k-1}} + m^2 \phi(t) + m]. \end{aligned}$$

□

Combining (3.6)-(3.7) with (3.5), we obtain

$$\Delta(\xi_t + \eta - F + \int F(x, t) dx) = 0.$$

Thus,

$$\xi_t + \eta - F + \int F(x, t) dx = 0.$$

Since $F = (2\mu + \lambda(\rho)) \operatorname{div} u - p - \frac{1}{2}|H|^2$, the preceding equality implies that

$$\xi_t - (2\mu + \lambda(\rho)) \operatorname{div} u + p + \frac{1}{2}|H|^2 + \eta + \int F(x, t) dx = 0.$$

Using the continuity equation, one gets

$$\xi_t + (2\mu + \lambda(\rho)) \frac{1}{\rho} (\rho_t + u \cdot \nabla \rho) + p(\rho) + \frac{1}{2}|H|^2 + \eta + \int F(x, t) dx = 0. \quad (3.11)$$

Define

$$\theta(\rho) = \int_1^\rho \frac{2\mu + \lambda(s)}{s} ds = 2\mu \ln \rho + \frac{1}{\beta} (\rho^\beta - 1),$$

then (3.11) yields the following transport equation

$$(\xi + \theta(\rho))_t + u \cdot \nabla (\xi + \theta(\rho)) + p(\rho) + \frac{1}{2}|H|^2 + \eta - u \cdot \nabla \xi + \int F(x, t) dx = 0. \quad (3.12)$$

Lemma 3.4. *For any $k \geq 1$, it holds that*

$$\sup_{t \in [0, T]} \|\rho(\cdot, t)\|_k \leq Ck^{\frac{2}{\beta-1}}.$$

Proof. Multiplying the equation (3.12) by the function $\rho[(\xi + \theta(\rho))_+]^{2m-1}$ with $m \geq 4$ a natural number, and integrating the result equation over Ω , involving the continuity equation, we get

$$\begin{aligned} & \frac{1}{2m} \frac{d}{dt} \int \rho[(\xi + \theta(\rho))_+]^{2m} dx + \int \rho p(\rho) [(\xi + \theta(\rho))_+]^{2m-1} dx \\ & + \frac{1}{2} \int \rho |H|^2 [(\xi + \theta(\rho))_+]^{2m-1} dx \\ & = - \int \rho \eta [(\xi + \theta(\rho))_+]^{2m-1} dx + \int \rho u \cdot \nabla \xi [(\xi + \theta(\rho))_+]^{2m-1} dx \\ & - \int F(x, t) dx \int \rho [(\xi + \theta(\rho))_+]^{2m-1} dx. \end{aligned} \tag{3.13}$$

Put

$$f(t) = \left\{ \int \rho [(\xi + \theta(\rho))_+]^{2m} \right\}^{\frac{1}{2m}}, \quad t \in [0, T].$$

Now we estimate the terms on the right-hand side of (3.13).

First of all,

$$\begin{aligned} & \left| - \int \rho \eta [(\xi + \theta(\rho))_+]^{2m-1} dx \right| \\ & \leq \int \rho^{\frac{1}{2m}} |\eta| [\rho(\xi + \theta(\rho))_+]^{\frac{2m-1}{2m}} dx \\ & \leq \|\rho\|_{2m\beta+1}^{\frac{1}{2m}} \|\eta\|_{2m+\frac{1}{\beta}} \|\rho(\xi + \theta(\rho))_+^{2m}\|_1^{\frac{2m-1}{2m}} \\ & \leq C \|\rho\|_{2m\beta+1}^{\frac{1}{2m}} f(t)^{2m-1} \left[\left(m + \frac{1}{2\beta}\right)^2 k \|\rho\|_{\frac{2(m+\frac{1}{2\beta})k}{k-1}} \|\nabla u\|_2^2 + m \|\rho\|_{\frac{2(m+\frac{1}{2\beta})k}{k-1}} \right. \\ & \quad \left. + m^2 \|\nabla H\|_2^2 + m \right] \\ & \leq C \|\rho\|_{2m\beta+1}^{1+\frac{1}{2m}} f(t)^{2m-1} [m^2 \phi(t) + m] + \|\rho\|_{2m\beta+1}^{\frac{1}{2m}} f(t)^{2m-1} [m^2 \phi(t) + m], \end{aligned} \tag{3.14}$$

where ϕ is defined as (3.8), and we have chosen $k = \frac{\beta}{\beta-1}$.

Next, for $\frac{1}{2m\beta+1} + \frac{2}{p} = 1$ with $p \geq 1$, the second term on the right-hand side of (3.13) can be estimated as

$$\begin{aligned}
& \left| \int \rho u \cdot \nabla \xi [(\xi + \theta(\rho))_+]^{2m-1} dx \right| \\
& \leq \int \rho^{\frac{1}{2m}} |u| |\nabla \xi| [\rho(\xi + \theta(\rho))_+]^{\frac{2m-1}{2m}} dx \\
& \leq C \|\rho\|_{2m\beta+1}^{\frac{1}{2m}} \|u\|_{2mp} \|\nabla \xi\|_{2mp} \|\rho(\xi + \theta(\rho))_+^{2m}\|_1^{\frac{2m-1}{2m}} \\
& \leq C \|\rho\|_{2m\beta+1}^{\frac{1}{2m}} [(mp)^{\frac{1}{2}} \|\nabla u\|_2 + 1] [(mq)^{\frac{3}{2}} k^{\frac{1}{2}} \|\rho\|_{\frac{2mqk}{k-1}}^{\frac{2m-1}{2}} \phi(t)^{\frac{1}{2}} \\
& \quad + mq \|\rho\|_{\frac{2mqk}{k-1}}] f(t)^{2m-1} \\
& \leq C \|\rho\|_{2m\beta+1}^{1+\frac{1}{2m}} f(t)^{2m-1} [m^{\frac{1}{2}} \phi(t)^{\frac{1}{2}} + 1] [m^{\frac{3}{2}} \phi(t)^{\frac{1}{2}} + m] \\
& \leq C \|\rho\|_{2m\beta+1}^{1+\frac{1}{2m}} f(t)^{2m-1} [m^2 \phi(t) + m],
\end{aligned} \tag{3.15}$$

where in the third inequality we have chosen $p = q = \frac{2m\beta+1}{m\beta}$ and $k = \frac{\beta}{\beta-2}$.

Now let's estimate the last term on the right-hand side of (3.13). From the elementary estimate (3.1), one can show that

$$\begin{aligned}
& \left| - \int F(x, t) dx \int \rho [(\xi + \theta(\rho))_+]^{2m-1} dx \right| \\
& \leq \int |(2\mu + \lambda(\rho)) \operatorname{div} u - p - \frac{1}{2} |H|^2| dx \int \rho^{\frac{1}{2m}} [\rho(\xi + \theta(\rho))_+]^{\frac{2m-1}{2m}} dx \\
& \leq \left[\left(\int (2\mu + \lambda(\rho)) (\operatorname{div} u)^2 dx \right)^{\frac{1}{2}} \left(\int (2\mu + \lambda(\rho)) dx \right)^{\frac{1}{2}} \right] + \int p(\rho) dx \\
& \quad + \frac{1}{2} \int |H|^2 dx \|\rho\|_1^{\frac{1}{2m}} \|\rho(\xi + \theta(\rho))_+^{2m}\|_1^{\frac{2m-1}{2m}} \\
& \leq C [\phi(t)^{\frac{1}{2}} + \phi(t)^{\frac{1}{2}} \left(\int \rho^\beta dx \right)^{\frac{1}{2}} + 1] f(t)^{2m-1} \\
& \leq C [\phi(t)^{\frac{1}{2}} + \phi(t)^{\frac{1}{2}} \|\rho\|_{2m\beta+1}^{\frac{\beta}{2}} + 1] f(t)^{2m-1}.
\end{aligned} \tag{3.16}$$

Plugging (3.14)-(3.16) into (3.13) yields that

$$\begin{aligned}
& \frac{1}{2m} \frac{d}{dt} (f^{2m}(t)) + \int \rho p(\rho) [(\xi + \theta(\rho))_+]^{2m-1} dx \\
& \leq C \|\rho\|_{2m\beta+1}^{1+\frac{1}{2m}} f(t)^{2m-1} [m^2 \phi(t) + m] + C \|\rho\|_{2m\beta+1}^{\frac{1}{2m}} f(t)^{2m-1} [m^2 \phi(t) + m] \\
& \quad + C \|\rho\|_{2m\beta+1}^{\frac{1}{2m}} [m^{\frac{1}{2}} \phi(t)^{\frac{1}{2}} + 1] f(t)^{2m-1} + C [\phi(t)^{\frac{1}{2}} + \phi(t)^{\frac{1}{2}} \|\rho\|_{2m\beta+1}^{\frac{\beta}{2}} + 1] f(t)^{2m-1}.
\end{aligned}$$

Then it holds that

$$\begin{aligned}
\frac{d}{dt} f(t) & \leq C [1 + \phi(t)^{\frac{1}{2}} + \phi(t)^{\frac{1}{2}} \|\rho\|_{2m\beta+1}^{\frac{\beta}{2}} + (m^2 \phi(t) + m) \|\rho\|_{2m\beta+1}^{1+\frac{1}{2m}} \\
& \quad + (m^2 \phi(t) + m) \|\rho\|_{2m\beta+1}^{\frac{1}{2m}} + (m^{\frac{1}{2}} \phi(t)^{\frac{1}{2}} + 1) \|\rho\|_{2m\beta+1}^{\frac{1}{2m}}].
\end{aligned}$$

Integrating the above inequality over $[0, t]$ gives that

$$f(t) \leq f(0) + C[1 + \int_0^t \phi(s)^{\frac{1}{2}} \|\rho\|_{2m\beta+1}^{\frac{\beta}{2}} ds + \int_0^t (m^2\phi(s) + m) \|\rho\|_{2m\beta+1}^{1+\frac{1}{2m}} ds] \quad (3.17)$$

$$+ \int_0^t (m^2\phi(s) + m) \|\rho\|_{2m\beta+1}^{\frac{1}{2m}} ds + \int_0^t (m^{\frac{1}{2}}\phi(s)^{\frac{1}{2}} + 1) \|\rho\|_{2m\beta+1}^{\frac{1}{2m}} ds]. \quad (3.18)$$

Now we calculate the quantity

$$f(0) = \left(\int \rho_0 [(\xi_0 + \theta(\rho_0))_+]^{2m} dx \right)^{\frac{1}{2m}}.$$

From Lemma 3.2(1), it is not difficulty to prove

$$\|\xi_0\|_{L^\infty} \leq C.$$

Moreover, from the definition of $\theta(\rho_0) = 2\mu \ln \rho_0 + \frac{1}{\beta}((\rho_0)^\beta - 1)$, one has

$$\xi_0 + \theta(\rho_0) \rightarrow -\infty, \text{ as } \rho_0 \rightarrow 0^+.$$

So there exists a positive constant σ , such that if $0 < \rho_0 \leq \sigma$, then

$$(\xi_0 + \theta(\rho_0))_+ \equiv 0.$$

Thus it holds that

$$\begin{aligned} f(0) &= \left[\left(\int_{[a \leq \rho_0 \leq \sigma]} + \int_{[\sigma \leq \rho_0 \leq M]} \right) \rho_0 (\xi_0 + \theta(\rho_0))_+^{2m} dx \right]^{\frac{1}{2m}} \\ &= \left[\int_{[\sigma \leq \rho_0 \leq M]} \rho_0 (\xi_0 + \theta(\rho_0))_+^{2m} dx \right]^{\frac{1}{2m}} \leq C_{\sigma, M}, \end{aligned}$$

with $C_{\sigma, M}$ a positive constant independent of m , which together with (3.17) leads to

$$\begin{aligned} f(t) &\leq C[1 + \int_0^t \phi(s)^{\frac{1}{2}} \|\rho\|_{2m\beta+1}^{\frac{\beta}{2}} ds + \int_0^t (m^2\phi(s) + m) \|\rho\|_{2m\beta+1}^{1+\frac{1}{2m}} ds \\ &\quad + \int_0^t (m^2\phi(s) + m) \|\rho\|_{2m\beta+1}^{\frac{1}{2m}} ds + \int_0^t (m^{\frac{1}{2}}\phi(s)^{\frac{1}{2}} + 1) \|\rho\|_{2m\beta+1}^{\frac{1}{2m}} ds]. \end{aligned} \quad (3.19)$$

Set $\Omega_1(t) = \{x \in \mathbb{T}^2 | \rho(x, t) > 2\}$ and $\Omega_2(t) = \{x \in \Omega_1(t) | \xi(x, t) + \theta(\rho)(x, t) > 0\}$. Then one has

$$\begin{aligned} \|\rho\|_{2m\beta+1}^\beta &= \left(\int \rho^{2m\beta+1} dx \right)^{\frac{\beta}{2m\beta+1}} = \left(\int_{\Omega_1(t)} \rho^{2m\beta+1} dx + \int_{\mathbb{T}^2/\Omega_1(t)} \rho^{2m\beta+1} dx \right)^{\frac{\beta}{2m\beta+1}} \\ &\leq C \left(\int_{\Omega_1(t)} \rho^{2m\beta+1} dx \right)^{\frac{\beta}{2m\beta+1}} + C \leq C \left(\int_{\Omega_1(t)} \rho |\theta(\rho)|^{2m} dx \right)^{\frac{\beta}{2m\beta+1}} + C \\ &\leq C \left(\int_{\Omega_2(t)} \rho |\theta(\rho) + \xi - \xi|^{2m} dx + \int_{\Omega_1(t)/\Omega_2(t)} \rho |\theta(\rho)|^{2m} dx \right)^{\frac{\beta}{2m\beta+1}} + C \\ &\leq C \left(\int_{\Omega_2(t)} \rho |\theta(\rho) + \xi|^{2m} dx + \int_{\Omega_2(t)} \rho |\xi|^{2m} dx + \int_{\Omega_1(t)/\Omega_2(t)} \rho |\xi|^{2m} dx \right)^{\frac{\beta}{2m\beta+1}} + C \\ &\leq C(f(t)^{2m} + \int_{\mathbb{T}^2} \rho |\xi|^{2m} dx)^{\frac{\beta}{2m\beta+1}} + C \leq C[f(t) + \left(\int_{\mathbb{T}^2} \rho |\xi|^{2m} dx \right)^{\frac{\beta}{2m\beta+1}} + 1]. \end{aligned}$$

Notice that

$$\begin{aligned}
\left(\int_{\mathbb{T}^2} \rho |\xi|^{2m} dx\right)^{\frac{\beta}{2m\beta+1}} &\lesssim \|\rho\|_{2m\beta+1}^{\frac{\beta}{2m\beta+1}} \|\xi\|^{2m} \|\xi\|^{\frac{\beta}{2m\beta+1}} \lesssim \|\rho\|_{2m\beta+1}^{\frac{\beta}{2m\beta+1}} \|\xi\|_{2m+\frac{1}{\beta}}^{\frac{2m\beta}{2m\beta+1}} \\
&\lesssim \|\rho\|_{2m\beta+1}^{\frac{\beta}{2m\beta+1}} \left[(m + \frac{1}{2\beta})^{\frac{1}{2}} \|\rho\|_{m+\frac{1}{2\beta}}^{\frac{1}{2}}\right]^{\frac{2m\beta}{2m\beta+1}} \\
&\lesssim m^{\frac{1}{2}} \|\rho\|_{2m\beta+1}^{\frac{\beta(m+1)}{2m\beta+1}}.
\end{aligned}$$

Then from Young inequality, it follows that

$$\begin{aligned}
\|\rho\|_{2m\beta+1}^{\beta} &= \left(\int \rho^{2m\beta+1} dx\right)^{\frac{\beta}{2m\beta+1}} \leq C[1 + f(t) + m^{\frac{1}{2}} \|\rho\|_{2m\beta+1}^{\frac{\beta(m+1)}{2m\beta+1}}] \\
&\leq \frac{1}{2} \|\rho\|_{2m\beta+1}^{\beta} + C(1 + f(t) + m^{\frac{m\beta+\frac{1}{2}}{m(2\beta-1)}}).
\end{aligned}$$

Thus one can get

$$\begin{aligned}
\|\rho\|_{2m\beta+1}^{\beta} &\leq C[f(t) + m^{\frac{\beta}{2\beta-1}}] \\
&\leq C[m^{\frac{\beta}{2\beta-1}} + \int_0^t \phi(s)^{\frac{1}{2}} \|\rho\|_{2m\beta+1}^{\frac{\beta}{2}} ds + \int_0^t (m^2 \phi(s) + m) \|\rho\|_{2m\beta+1}^{1+\frac{1}{2m}} ds \\
&\quad + \int_0^t (m^2 \phi(s) + m) \|\rho\|_{2m\beta+1}^{\frac{1}{2m}} ds + \int_0^t (m^{\frac{1}{2}} \phi(s)^{\frac{1}{2}} + 1) \|\rho\|_{2m\beta+1}^{\frac{1}{2m}} ds] \\
&\leq C[m^{\frac{\beta}{2\beta-1}} + \int_0^t \|\rho\|_{2m\beta+1}^{\beta} ds + \int_0^t (m^2 \phi(s) + m) \|\rho\|_{2m\beta+1}^{1+\frac{1}{2m}} ds \\
&\quad + \int_0^t (m^2 \phi(s) + m) \|\rho\|_{2m\beta+1}^{\frac{1}{2m}} ds + \int_0^t (m^{\frac{1}{2}} \phi(s)^{\frac{1}{2}} + 1) \|\rho\|_{2m\beta+1}^{\frac{1}{2m}} ds].
\end{aligned}$$

From Gronwall's inequality, it can be derived as

$$\begin{aligned}
\|\rho\|_{2m\beta+1}^{\beta} &\leq C[m^{\frac{\beta}{2\beta-1}} + \int_0^t (m^2 \phi(s) + m) \|\rho\|_{2m\beta+1}^{1+\frac{1}{2m}} ds + \int_0^t (m^2 \phi(s) + m) \|\rho\|_{2m\beta+1}^{\frac{1}{2m}} ds \\
&\quad + \int_0^t (m^{\frac{1}{2}} \phi(s)^{\frac{1}{2}} + 1) \|\rho\|_{2m\beta+1}^{\frac{1}{2m}} ds].
\end{aligned}$$

Put

$$y(t) = m^{-\frac{2}{\beta-1}} \|\rho\|_{2m\beta+1}(t),$$

then

$$\begin{aligned}
y^{\beta}(t) &\leq C[m^{\frac{\beta(1-3\beta)}{(2\beta-1)(\beta-1)}} + m^{\frac{1}{m(\beta-1)}} \int_0^t \phi(s) y(s)^{1+\frac{1}{2m}} ds \\
&\quad + m^{\frac{1}{m(\beta-1)-1}} \int_0^t y(s)^{1+\frac{1}{2m}} ds + m^{\frac{1-2m}{m(\beta-1)}} \int_0^t \phi(s) y(s)^{\frac{1}{2m}} ds \\
&\quad + m^{\frac{1-m-m\beta}{m(\beta-1)}} \int_0^t y(s)^{\frac{1}{2m}} ds + m^{\frac{1-2m}{m(\beta-1)}-\frac{3}{2}} \int_0^t \phi(s)^{\frac{1}{2}} y(s)^{\frac{1}{2m}} ds \\
&\quad + m^{\frac{1-2m\beta}{m(\beta-1)}} \int_0^t y(s)^{\frac{1}{2m}} ds] \\
&\leq C[1 + \int_0^t (\phi(s) + 1) y^{\beta}(s) ds].
\end{aligned}$$

Again applying Gronwall's inequality, we have

$$y(t) \leq C, \quad \forall t \in [0, T],$$

which implies

$$\|\rho\|_{2m\beta+1}(t) \leq Cm^{\frac{2}{\beta-1}}, \quad \forall t \in [0, T].$$

□

Step 3. First-order derivative estimates of the velocity u and the magnetic field H .

Lemma 3.5. *There exists a positive constant C , such that*

$$\sup_{t \in [0, T]} \int \mu \omega^2 + (\operatorname{curl} H)^2 + \frac{(F + \frac{1}{2}|H|^2)^2}{2\mu + \lambda(\rho)} dx + \int_0^T \int \rho(B^2 + L^2) + \nu |\nabla \operatorname{curl} H|^2 dx dt \leq C.$$

Proof. Multiplying the first equation and the second equation of (2.3) by $\mu \omega$ and $\frac{(F + \frac{1}{2}|H|^2)}{2\mu + \lambda(\rho)}$ respectively, and then summing the resulted equations together, one has

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \mu \omega + \frac{(F + \frac{1}{2}|H|^2)^2}{2\mu + \lambda(\rho)} dx + \int \rho(B^2 + L^2) dx \\ &= -\frac{\mu}{2} \int \omega^2 \operatorname{div} u dx - \left(\frac{1}{\rho} H \cdot \nabla H_1\right)_{x_2} \cdot \omega + \left(\frac{1}{\rho} H \cdot \nabla H_1\right)_{x_1} \cdot \omega dx \\ &+ \frac{1}{2} \int (F + \frac{1}{2}|H|^2)^2 \operatorname{div} u \left[\rho \left(\frac{1}{2\mu + \lambda(\rho)}\right)' - \frac{1}{2\mu + \lambda(\rho)}\right] dx \\ &+ \int (F + \frac{1}{2}|H|^2) \operatorname{div} u \left[\rho \left(\frac{p}{2\mu + \lambda(\rho)}\right)' - \frac{p}{2\mu + \lambda(\rho)}\right] dx \\ &- \int 2(F + \frac{1}{2}|H|^2)(u_{1x_2} u_{2x_1} - u_{1x_1} u_{2x_2}) dx \\ &- \int \frac{2}{\rho} (F + \frac{1}{2}|H|^2)(H_{1x_2} H_{2x_1} - H_{1x_1} H_{2x_2}) dx \\ &+ \int (F + \frac{1}{2}|H|^2) \left[\left(\frac{1}{\rho}\right)_{x_1} H \cdot \nabla H_1 + \left(\frac{1}{\rho}\right)_{x_2} H \cdot \nabla H_2\right] dx \\ &+ \int B \left(\frac{1}{2}|H|^2\right)_{x_2} + L \left(\frac{1}{2}|H|^2\right)_{x_1} dx, \end{aligned} \tag{3.20}$$

where we have used the fact that

$$\begin{aligned} & (u_{1x_1})^2 + 2u_{1x_2} u_{2x_1} + (u_{2x_2})^2 \\ &= (u_{1x_1} + u_{2x_2})^2 + 2(u_{1x_2} u_{2x_1} - u_{1x_1} u_{2x_2}) \\ &= (\operatorname{div} u)^2 + 2(u_{1x_2} u_{2x_1} - u_{1x_1} u_{2x_2}) \\ &= \operatorname{div} u \left(\frac{F + \frac{1}{2}|H|^2 + p(\rho)}{2\mu + \lambda(\rho)}\right) + 2(u_{1x_2} u_{2x_1} - u_{1x_1} u_{2x_2}) \end{aligned}$$

and

$$\begin{aligned} & (H_{1x_1})^2 + 2H_{1x_2} H_{2x_1} + (H_{2x_2})^2 \\ &= (H_{1x_1} + H_{2x_2})^2 + 2(H_{1x_2} H_{2x_1} - H_{1x_1} H_{2x_2}) \\ &= 2(H_{1x_2} H_{2x_1} - H_{1x_1} H_{2x_2}). \end{aligned}$$

Applying the operator curl to the magnetic equation of (2.4), multiplying the resulting equation by $\text{curl}H$, then integrating it over \mathbb{T}^2 , we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\text{curl}H)^2 dx + \nu \int |\nabla \text{curl}|^2 dx - \frac{1}{2} \int (\text{curl}H)^2 \text{div}u dx \\ & + \int u_{x_1} \cdot \nabla H_{x_2} \cdot \text{curl}H dx - \int u_{x_2} \cdot \nabla H_{x_1} \cdot \text{curl}H dx \\ & + \int H \cdot \nabla u_2 + H_2 \text{div}u(\text{curl}H)_{x_1} dx - \int H \cdot \nabla u_1 + H_1 \text{div}u(\text{curl}H)_{x_2} dx, \end{aligned}$$

which together with (3.20) gives that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \mu\omega + (\text{curl}H)^2 + \frac{(F + \frac{1}{2}|H|^2)^2}{2\mu + \lambda(\rho)} dx + \int \rho(B^2 + L^2) + \nu |\nabla \text{curl}|^2 dx \\ & = -\frac{\mu}{2} \int \omega^2 \text{div}u dx - \left(\frac{1}{\rho} H \cdot \nabla H_1\right)_{x_2} \cdot \omega + \left(\frac{1}{\rho} H \cdot \nabla H_1\right)_{x_1} \cdot \omega dx \\ & + \frac{1}{2} \int (F + \frac{1}{2}|H|^2)^2 \text{div}u \left[\rho \left(\frac{1}{2\mu + \lambda(\rho)}\right)' - \frac{1}{2\mu + \lambda(\rho)}\right] dx \\ & + \int (F + \frac{1}{2}|H|^2) \text{div}u \left[\rho \left(\frac{p}{2\mu + \lambda(\rho)}\right)' - \frac{p}{2\mu + \lambda(\rho)}\right] dx \\ & - \int 2(F + \frac{1}{2}|H|^2)(u_{1x_2}u_{2x_1} - u_{1x_1}u_{2x_2}) dx \\ & - \int \frac{2}{\rho} (F + \frac{1}{2}|H|^2)(H_{1x_2}H_{2x_1} - H_{1x_1}H_{2x_2}) dx \\ & + \int (F + \frac{1}{2}|H|^2) \left[\left(\frac{1}{\rho}\right)_{x_1} H \cdot \nabla H_1 + \left(\frac{1}{\rho}\right)_{x_2} H \cdot \nabla H_2\right] dx \\ & + \int B \left(\frac{1}{2}|H|^2\right)_{x_2} + L \left(\frac{1}{2}|H|^2\right)_{x_1} dx - \frac{1}{2} \int (\text{curl}H)^2 \text{div}u dx \\ & - \int u_{x_1} \cdot \nabla H_{x_2} \cdot \text{curl}H dx + \int u_{x_2} \cdot \nabla H_{x_1} \cdot \text{curl}H dx \\ & - \int H \cdot \nabla u_2 + H_2 \text{div}u(\text{curl}H)_{x_1} dx + \int H \cdot \nabla u_1 + H_1 \text{div}u(\text{curl}H)_{x_2} dx. \end{aligned} \tag{3.21}$$

Denote

$$Z^2(t) = \int \mu\omega + (\text{curl}H)^2 + \frac{(F + \frac{1}{2}|H|^2)^2}{2\mu + \lambda(\rho)} dx \tag{3.22}$$

and

$$\varphi^2(t) = \int \rho(B^2 + L^2) + \nu |\nabla \text{curl}|^2 dx. \tag{3.23}$$

Then it holds that for $0 < r \leq \frac{1}{2}$,

$$\begin{aligned} \|\nabla(F + |H|^2), \omega, \text{curl}H\|_{2(1-r)} & \leq C\varphi(t) \|\rho\|^{\frac{1}{2}}_{\frac{1-r}{r}} \leq C\varphi(t) \left(\frac{1-r}{r}\right)^{\frac{1}{\beta-1}} \\ & \leq C\varphi(t) r^{\frac{1}{1-\beta}} \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} \|\nabla u\|_2 + \|\omega\|_2 + \|(2\mu + \lambda(\rho))^{\frac{1}{2}} \operatorname{div} u\|_2 &\leq C[Z(t) + (\int \frac{p(\rho)^2}{2\mu + \lambda(\rho)} dx)^{\frac{1}{2}}] \\ &\leq C(Z(t) + 1). \end{aligned} \quad (3.25)$$

Now let's estimate the terms on the right-hand side of (3.21). From the Hölder inequality, interpolation inequality, the elementary estimate (3.1), (3.24)-(3.25), it follows that for $0 < \varepsilon \leq \frac{1}{4}$,

$$\begin{aligned} -\frac{\mu}{2} \int \omega^2 \operatorname{div} u dx &\leq C \|\operatorname{div} u\|_2 \|\omega\|_4^2 \leq C(Z(t) + 1) \|\omega\|_2^{\frac{1-3\varepsilon}{1-2\varepsilon}} \|\nabla \omega\|_{2(1-\varepsilon)}^{\frac{1-\varepsilon}{1-2\varepsilon}} \\ &\leq C(Z(t) + 1) Z(t)^{\frac{1-3\varepsilon}{1-2\varepsilon}} \varphi(t)^{\frac{1-\varepsilon}{1-2\varepsilon} \varepsilon^{\frac{1-\varepsilon}{(1-\beta)(1-2\varepsilon)}}} \\ &\leq \delta \varphi^2(t) + C_\delta Z^2(t) (Z(t) + 1)^{\frac{2(1-2\varepsilon)}{1-3\varepsilon} \varepsilon^{\frac{2(1-\varepsilon)}{(1-\beta)(1-3\varepsilon)}}} \\ &\leq \delta \varphi^2(t) + C_\delta (Z^2(t) + 1)^{2 + \frac{\varepsilon}{1-3\varepsilon} \varepsilon^{\frac{2(1-\varepsilon)}{(1-\beta)(1-3\varepsilon)}}}, \end{aligned} \quad (3.26)$$

where and in the sequel $\delta > 0$ is a small positive constant to be determined and C_δ is a positive constant depending on δ .

From the definition of F and $\lambda(\rho)$, and Lemma 2.4, similarly one has

$$\begin{aligned} -(\frac{1}{\rho} H \cdot \nabla H_1)_{x_2} \cdot \omega + (\frac{1}{\rho} H \cdot \nabla H_1)_{x_1} \cdot \omega dx &\leq C \|\nabla \omega\|_2 \|\nabla H\|_2 \|H\|_{L^\infty} \\ &\leq C \varphi(t) \|\operatorname{curl} H\|_2 \leq \delta \varphi^2(t) + C_\delta (Z(t)^2 + 1), \end{aligned} \quad (3.27)$$

$$\begin{aligned} &\frac{1}{2} \int (F + \frac{1}{2} |H|^2)^2 \operatorname{div} u [\rho (\frac{1}{2\mu + \lambda(\rho)})' - \frac{1}{2\mu + \lambda(\rho)}] dx \\ &= \frac{1}{2} \int (F + \frac{1}{2} |H|^2)^2 (\frac{F + \frac{1}{2} |H|^2}{2\mu + \lambda(\rho)} + \frac{p(\rho)}{2\mu + \lambda(\rho)}) \frac{2\mu + \lambda(\rho) + \rho \lambda'(\rho)}{(2\mu + \lambda(\rho))^2} dx \\ &\leq C \int (F + \frac{1}{2} |H|^2)^2 (\frac{F + \frac{1}{2} |H|^2}{2\mu + \lambda(\rho)} + \frac{p(\rho)}{2\mu + \lambda(\rho)}) dx \\ &\leq C(1 + \int \frac{|F + \frac{1}{2} |H|^2|^3}{2\mu + \lambda(\rho)} dx), \end{aligned} \quad (3.28)$$

$$\begin{aligned} &\int (F + \frac{1}{2} |H|^2) \operatorname{div} u [\rho (\frac{p}{2\mu + \lambda(\rho)})' - \frac{p}{2\mu + \lambda(\rho)}] dx \\ &= \int (F + \frac{1}{2} |H|^2) (\frac{F + \frac{1}{2} |H|^2}{2\mu + \lambda(\rho)} + \frac{p(\rho)}{2\mu + \lambda(\rho)}) \\ &\quad \times \frac{p(\rho)(2\mu + \lambda(\rho)) + \rho \lambda'(\rho) p(\rho) - \rho p'(\rho)(2\mu + \lambda(\rho))}{(2\mu + \lambda(\rho))^2} dx \\ &\leq C \int (F + \frac{1}{2} |H|^2) (\frac{F + \frac{1}{2} |H|^2}{2\mu + \lambda(\rho)} + \frac{p(\rho)}{2\mu + \lambda(\rho)}) p(\rho) dx \\ &\leq C(1 + \int \frac{|F + \frac{1}{2} |H|^2|^3}{2\mu + \lambda(\rho)} dx), \end{aligned} \quad (3.29)$$

$$- \int 2(F + \frac{1}{2} |H|^2) (u_{1x_2} u_{2x_1} - u_{1x_1} u_{2x_2}) dx \leq C \int |F + \frac{1}{2} |H|^2| |\nabla u|^2 dx, \quad (3.30)$$

$$\begin{aligned}
& - \int \frac{2}{\rho} (F + \frac{1}{2}|H|^2)(H_{1x_2}H_{2x_1} - H_{1x_1}H_{2x_2})dx \\
& \leq C \int |F + \frac{1}{2}|H|^2||\nabla H|^2 dx,
\end{aligned} \tag{3.31}$$

□

$$\begin{aligned}
& \int B(\frac{1}{2}|H|^2)_{x_2} + L(\frac{1}{2}|H|^2)_{x_1} dx \\
& \leq C(\|\nabla H\|_2 \|H\|_{L^\infty} \|\sqrt{\rho}B\|_2 + \|\nabla H\|_2 \|H\|_{L^\infty} \|\sqrt{\rho}L\|_2) \\
& \leq C(Z(t) + 1)\varphi(t) \leq \delta\varphi^2(t) + C_\delta(Z^2(t) + 1),
\end{aligned} \tag{3.32}$$

$$\begin{aligned}
& - \frac{1}{2} \int (\text{curl}H)^2 \text{div}u dx \leq C\|\text{div}u\|_2 \|\text{curl}H\|_4^2 \\
& \leq C(Z(t) + 1) \|\text{curl}H\|_2^{\frac{1-3\varepsilon}{1-2\varepsilon}} \|\nabla \text{curl}H\|_{2(1-\varepsilon)}^{\frac{1-\varepsilon}{1-2\varepsilon}} \\
& \leq C(Z(t) + 1) Z(t)^{\frac{1-3\varepsilon}{1-2\varepsilon}} \varphi(t)^{\frac{1-\varepsilon}{1-2\varepsilon}} \varepsilon^{\frac{1-\varepsilon}{(1-\beta)(1-2\varepsilon)}} \\
& \leq \delta\varphi^2(t) + C_\delta Z^2(t)(Z(t) + 1)^{\frac{2(1-2\varepsilon)}{1-3\varepsilon}} \varepsilon^{\frac{2(1-\varepsilon)}{(1-\beta)(1-3\varepsilon)}} \\
& \leq \delta\varphi^2(t) + C_\delta(Z(t)^2 + 1)^{2+\frac{\varepsilon}{1-3\varepsilon}} \varepsilon^{\frac{2(1-\varepsilon)}{(1-\beta)(1-3\varepsilon)}},
\end{aligned} \tag{3.33}$$

$$\begin{aligned}
& - \int u_{x_1} \cdot \nabla H_{x_2} \cdot \text{curl}H dx + \int u_{x_2} \cdot \nabla H_{x_1} \cdot \text{curl}H dx \\
& \leq C\|\nabla H\|_4^2 \|\nabla u\|_2 \leq C(Z(t) + 1) \|\text{curl}H\|_4^2 \\
& \leq C(Z(t) + 1) \|\text{curl}H\|_2^{\frac{1-3\varepsilon}{1-2\varepsilon}} \|\nabla \text{curl}H\|_{2(1-\varepsilon)}^{\frac{1-\varepsilon}{1-2\varepsilon}} \\
& \leq \delta\varphi^2(t) + C_\delta(Z(t)^2 + 1)^{2+\frac{\varepsilon}{1-3\varepsilon}} \varepsilon^{\frac{2(1-\varepsilon)}{(1-\beta)(1-3\varepsilon)}}
\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}
& - \int H \cdot \nabla u_2 + H_2 \text{div}u(\text{curl}H)_{x_1} dx + \int H \cdot \nabla u_1 + H_1 \text{div}u(\text{curl}H)_{x_2} dx \\
& \leq \|\nabla \text{curl}H\|_2 \|\nabla u\|_2 \|H\|_{L^\infty} \leq C\varphi(t)(Z(t) + 1) \\
& \leq \delta\varphi^2(t) + C_\delta(Z^2(t) + 1).
\end{aligned} \tag{3.35}$$

Substituting (3.28)-(3.35) into (3.26) leads to

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} Z^2(t) + \varphi^2(t) \leq \delta\varphi^2(t) + C_\delta(1 + Z^2(t) + (Z^2(t) + 1)^{2+\frac{\varepsilon}{1-3\varepsilon}} \varepsilon^{\frac{2(1-\varepsilon)}{(1-\beta)(1-3\varepsilon)}}) \\
& + C[1 + \int \frac{|F + \frac{1}{2}|H|^2|^3}{2\mu + \lambda(\rho)} dx + \int |F + \frac{1}{2}|H|^2||\nabla u|^2 dx \\
& + \int |F + \frac{1}{2}|H|^2||\nabla H|^2 dx].
\end{aligned} \tag{3.36}$$

Now it remains to estimate the terms $\int \frac{|F + \frac{1}{2}|H|^2|^3}{2\mu + \lambda(\rho)} dx$, $\int |F + \frac{1}{2}|H|^2||\nabla u|^2 dx$ and $\int |F + \frac{1}{2}|H|^2||\nabla H|^2 dx$ on the right-hand side of (3.36).

From Lemma 2.3, for $\varepsilon \in [0, \frac{1}{2}]$ and $\eta = \varepsilon$, it follows that

$$\|F + \frac{1}{2}|H|^2\|_{2m} \leq C[\|F + \frac{1}{2}|H|^2\|_1 + m^{\frac{1}{2}}\|\nabla(F + \frac{1}{2}|H|^2)\|_{\frac{2m}{m+\varepsilon}}^{1-s}\|F + \frac{1}{2}|H|^2\|_{2(1-\varepsilon)}^s], \quad (3.37)$$

where $s = \frac{(1-\varepsilon)^2}{m-\varepsilon(1-\varepsilon)}$ and the positive constant C is independent of m and ε . Choose the positive constant $\varepsilon = 2^{-m}$ with $m > 2$ being integer in the inequalities (3.36) and (3.37). From Lemma 3.4 one can get

$$\begin{aligned} \|F + \frac{1}{2}|H|^2\|_1 &= \int (2\mu + \lambda(\rho))^{-\frac{1}{2}} |F + \frac{1}{2}|H|^2| (2\mu + \lambda(\rho))^{\frac{1}{2}} dx \\ &\leq \left(\int \frac{(F + \frac{1}{2}|H|^2)^2}{2\mu + \lambda(\rho)} dx \right)^{\frac{1}{2}} \left(\int 2\mu + \lambda(\rho) dx \right)^{\frac{1}{2}} \leq CZ(t), \end{aligned} \quad (3.38)$$

$$\begin{aligned} &\|F + \frac{1}{2}|H|^2\|_{2(1-\varepsilon)}^s \\ &= \left(\int (2\mu + \lambda(\rho))^{-(1-\varepsilon)} |F + \frac{1}{2}|H|^2|^{2(1-\varepsilon)} (2\mu + \lambda(\rho))^{1-\varepsilon} dx \right)^{\frac{s}{2(1-\varepsilon)}} \\ &\leq \left(\int \frac{(F + \frac{1}{2}|H|^2)^2}{2\mu + \lambda(\rho)} dx \right)^{\frac{s}{2}} \left(\int (2\mu + \lambda(\rho))^{\frac{1-\varepsilon}{2}} dx \right)^{\frac{s\varepsilon}{2(1-\varepsilon)}} \\ &\leq CZ^s(t) \left(\|\rho\|^{\frac{s\beta}{2(1-\varepsilon)}} + 1 \right) \leq CZ^s(t) \left[\left(\frac{\beta(1-\varepsilon)}{\varepsilon} \right)^{\frac{s\beta}{\beta-1}} + 1 \right] \\ &\leq CZ^s(t) \left[(\varepsilon^{-\frac{s\beta}{\beta-1}} + 1) \right] \leq CZ^s(t) (2^{\frac{ms\beta}{\beta-1}} + 1) \leq CZ^s(t), \end{aligned} \quad (3.39)$$

where in the last inequality one has used the fact that $ms = \frac{m(1-\varepsilon)^2}{m-\varepsilon(1-\varepsilon)} \rightarrow 1$, as $m \rightarrow \infty$.

Inserting (3.24) with $r = \frac{\varepsilon}{m+\varepsilon}$, (3.38)-(3.39) into (3.37) implies that

$$\begin{aligned} \|F + \frac{1}{2}|H|^2\|_{2m} &\leq C[Z(t) + m^{\frac{1}{2}}\|\nabla(F + \frac{1}{2}|H|^2)\|_{\frac{2m}{m+\varepsilon}}^{1-s}Z(t)^s] \\ &\leq C[Z(t) + m^{\frac{1}{2}}\left(\frac{m+\varepsilon}{\varepsilon}\right)^{\frac{1-s}{\beta-1}}\varphi(t)^{1-s}Z(t)^s] \\ &\leq C[Z(t) + m^{\frac{1}{2}}\left(\frac{m}{\varepsilon}\right)^{\frac{1-s}{\beta-1}}\varphi(t)^{1-s}Z(t)^s]. \end{aligned} \quad (3.40)$$

Therefore, it holds that

$$\begin{aligned}
& \int \frac{|F + \frac{1}{2}|H|^2|^3}{2\mu + \lambda(\rho)} dx \\
&= \int \frac{|F + \frac{1}{2}|H|^2|^{2-\frac{1}{m-1}}}{(2\mu + \lambda(\rho))^{1-\frac{1}{2(m-1)}}} \left(\frac{1}{2\mu + \lambda(\rho)}\right)^{\frac{1}{2(m-1)}} |F + \frac{1}{2}|H|^2|^{1+\frac{1}{m-1}} dx \\
&\leq \int \left(\frac{|F + \frac{1}{2}|H|^2|^2}{2\mu + \lambda(\rho)}\right)^{1-\frac{1}{2(m-1)}} |F + \frac{1}{2}|H|^2|^{\frac{m}{m-1}} dx \\
&\leq \left(\int \frac{|F + \frac{1}{2}|H|^2|^2}{2\mu + \lambda(\rho)} dx\right)^{\frac{2m-3}{2(m-1)}} \left(\int |F + \frac{1}{2}|H|^2|^{2m} dx\right)^{\frac{1}{2(m-1)}} \\
&\leq Z(t)^{\frac{2m-3}{m-1}} \|F + \frac{1}{2}|H|^2\|_{2m}^{\frac{m}{m-1}} \\
&\leq CZ(t)^{\frac{2m-3}{m-1}} [Z(t) + m^{\frac{1}{2}} \left(\frac{m}{\varepsilon}\right)^{\frac{1-s}{\beta-1}} \varphi(t)^{1-s} Z(t)^s]^{\frac{m}{m-1}} \\
&\leq C[Z(t)^3 + m^{\frac{m}{2(m-1)}} \left(\frac{m}{\varepsilon}\right)^{\frac{(1-s)m}{(\beta-1)(m-1)}} \varphi(t)^{\frac{(1-s)m}{m-1}} Z(t)^{\frac{(2+s)m-3}{m-1}}] \\
&\leq \delta \varphi^2(t) + C_\delta [Z(t)^3 + m^{\frac{m}{m(1+s)-2}} \left(\frac{m}{\varepsilon}\right)^{\frac{2(1-s)m}{(\beta-1)(m(1+s)-2)}} Z(t)^{\frac{2((2+s)m-3)}{m(1+s)-2}}] \\
&\leq \delta \varphi^2(t) + C_\delta [(1 + Z^2(t))^2 + m \left(\frac{m}{\varepsilon}\right)^{\frac{2}{\beta-1}} (1 + Z^2(t))^{2+\frac{1-ms}{m(1+s)-2}}],
\end{aligned} \tag{3.41}$$

where we have applied the fact

$$ms = \frac{m(1-\varepsilon)^2}{m-\varepsilon(1-\varepsilon)} \rightarrow 1 \text{ as } m \rightarrow +\infty$$

and

$$\lim_{m \rightarrow +\infty} (2^m(1-ms)) = 2,$$

with $\varepsilon = 2^{-m}$.

Now consider the integral

$$\begin{aligned}
& \int |F + \frac{1}{2}|H|^2| |\nabla u|^2 dx \leq \|F + \frac{1}{2}|H|^2\|_{2m} \|\nabla u\|_{\frac{4m}{2m-1}}^2 \\
&\leq C \|F + \frac{1}{2}|H|^2\|_{2m} (\|\operatorname{div} u\|_{\frac{4m}{2m-1}}^2 + \|\omega\|_{\frac{4m}{2m-1}}^2) \\
&\leq C \|F + \frac{1}{2}|H|^2\|_{2m} \left(\left\|\frac{F + \frac{1}{2}|H|^2}{2\mu + \lambda(\rho)}\right\|_{\frac{4m}{2m-1}}^2 + \|\omega\|_{\frac{4m}{2m-1}}^2 + 1\right).
\end{aligned} \tag{3.42}$$

Note that

$$\begin{aligned}
& \left\| \frac{F + \frac{1}{2}|H|^2}{2\mu + \lambda(\rho)} \right\|_{\frac{4m}{2m-1}}^2 = \left(\int \frac{|F + \frac{1}{2}|H|^2|^{\frac{4m}{2m-1}}}{(2\mu + \lambda(\rho))^{\frac{4m}{2m-1}}} dx \right)^{\frac{2m-1}{2m}} \\
& = \left(\int \frac{|F + \frac{1}{2}|H|^2|^{\frac{2m(2m-3)}{(2m-1)(m-1)}}}{(2\mu + \lambda(\rho))^{\frac{4m}{2m-1}}} |F + \frac{1}{2}|H|^2|^{\frac{2m}{(2m-1)(m-1)}} dx \right)^{\frac{2m-1}{2m}} \\
& \leq \|F + \frac{1}{2}|H|^2\|_{\frac{4m}{2m-1}}^{\frac{1}{m-1}} \left(\int \frac{|F + \frac{1}{2}|H|^2|^2}{(2\mu + \lambda(\rho))^{\frac{4(m-1)}{2m-3}}} dx \right)^{\frac{2m-3}{2(m-1)}} \\
& \leq C \|F + \frac{1}{2}|H|^2\|_{\frac{4m}{2m-1}}^{\frac{1}{m-1}} \left(\int \frac{|F + \frac{1}{2}|H|^2|^2}{2\mu + \lambda(\rho)} dx \right)^{\frac{2m-3}{2(m-1)}} \\
& \leq C \|F + \frac{1}{2}|H|^2\|_{\frac{4m}{2m-1}}^{\frac{1}{m-1}} Z(t)^{\frac{2m-3}{2(m-1)}}
\end{aligned} \tag{3.43}$$

and by virtue of $\int \omega dx = 0$, interpolation inequality and (3.24), it can be derived as

$$\begin{aligned}
\|\omega\|_{\frac{4m}{2m-1}}^2 & \leq C \|\omega\|_2^{2-\frac{1-\varepsilon}{m(1-2\varepsilon)}} \|\nabla \omega\|_{\frac{4m}{2(1-\varepsilon)}}^{\frac{1-\varepsilon}{m(1-2\varepsilon)}} \\
& \leq CZ(t)^{2-\frac{1-\varepsilon}{m(1-2\varepsilon)}} (\varphi(t)\varepsilon^{\frac{1}{1-\beta}})^{\frac{1-\varepsilon}{m(1-2\varepsilon)}} \\
& \leq CZ(t)^{2-\frac{1-\varepsilon}{m(1-2\varepsilon)}} 2^{\frac{1-\varepsilon}{(\beta-1)(1-2\varepsilon)}} \varphi(t)^{\frac{1-\varepsilon}{m(1-2\varepsilon)}} \\
& \leq CZ(t)^{2-\frac{1-\varepsilon}{m(1-2\varepsilon)}} \varphi(t)^{\frac{1-\varepsilon}{m(1-2\varepsilon)}},
\end{aligned} \tag{3.44}$$

which together with (3.40), (3.42), (3.43) implies that

$$\begin{aligned}
& \int |F + \frac{1}{2}|H|^2| |\nabla u|^2 dx \\
& \leq C[Z(t) + m^{\frac{1}{2}}(\frac{m}{\varepsilon})^{\frac{1-s}{\beta-1}} \varphi(t)^{1-s} Z(t)^s]^{1+\frac{1}{m-1}} Z(t)^{\frac{2m-3}{2(m-1)}} \\
& + C[Z(t) + m^{\frac{1}{2}}(\frac{m}{\varepsilon})^{\frac{1-s}{\beta-1}} \varphi(t)^{1-s} Z(t)^s] [1 + Z(t)^{2-\frac{1-\varepsilon}{m(1-2\varepsilon)}} \varphi(t)^{\frac{1-\varepsilon}{m(1-2\varepsilon)}}] \\
& \leq C[Z(t)^3 + m^{\frac{m}{2(m-1)}} (\frac{m}{\varepsilon})^{\frac{m(1-s)}{(\beta-1)(m-1)}} \varphi(t)^{\frac{m(1-s)}{m-1}} Z(t)^{\frac{ms+2m-3}{m-1}} + Z(t) \\
& + Z(t)^{3-\frac{1-\varepsilon}{m(1-2\varepsilon)}} \varphi(t)^{\frac{1-\varepsilon}{m(1-2\varepsilon)}} + m^{\frac{1}{2}}(\frac{m}{\varepsilon})^{\frac{1-s}{\beta-1}} \varphi(t)^{1-s} Z(t)^s \\
& + m^{\frac{1}{2}}(\frac{m}{\varepsilon})^{\frac{1-s}{\beta-1}} \varphi(t)^{1-s+\frac{1-\varepsilon}{m(1-2\varepsilon)}} Z(t)^{2+s-\frac{1-\varepsilon}{m(1-2\varepsilon)}}] \\
& \leq \delta \varphi^2(t) + C_\delta [(1 + Z^2(t))^2 + (m^{\frac{1}{2}}(\frac{m}{\varepsilon})^{\frac{1-s}{\beta-1}} Z^s(t))^{\frac{2}{1+s}} \\
& + (m^{\frac{m}{2(m-1)}} (\frac{m}{\varepsilon})^{\frac{m(1-s)}{(\beta-1)(m-1)}} Z(t)^{2+\frac{ms-1}{m-1}})^{\frac{2(m-1)}{m(1+s)-2}} \\
& + (m^{\frac{1}{2}}(\frac{m}{\varepsilon})^{\frac{1-s}{\beta-1}} Z(t)^{2+s-\frac{1-\varepsilon}{m(1-2\varepsilon)}})^{\frac{2}{1+s-\frac{1-\varepsilon}{m(1-2\varepsilon)}}] \\
& \leq \delta \varphi^2(t) + C_\delta [(1 + Z^2(t))^2 + m(\frac{m}{\varepsilon})^{\frac{2}{\beta-1}} (1 + Z^2(t))^{2+\frac{1-ms}{m(1+s)-2}} \\
& + m(\frac{m}{\varepsilon})^{\frac{2}{\beta-1}} (1 + Z^2(t)) + m(\frac{m}{\varepsilon})^{\frac{2}{\beta-1}} (1 + Z^2(t))^{2+\frac{1-ms+(2ms-1)\varepsilon}{m(1+s)(1-2\varepsilon)-1+\varepsilon}}].
\end{aligned} \tag{3.45}$$

By $\int \operatorname{curl} H dx = 0$, Lemma 2.2, (3.24) and (3.40), after a similar estimate we arrive at

$$\begin{aligned}
& \int |F + \frac{1}{2}|H|^2| |\nabla H|^2 dx \\
& \leq \|F + \frac{1}{2}|H|^2\|_{2m} \|\nabla H\|_{\frac{4m}{2m-1}}^2 \leq C \|F + \frac{1}{2}|H|^2\|_{2m} \|\operatorname{curl} H\|_{\frac{4m}{2m-1}}^2 \\
& \leq C \|F + \frac{1}{2}|H|^2\|_{2m} \|\operatorname{curl} H\|_2^{2-\frac{1-\varepsilon}{m(1-2\varepsilon)}} \|\nabla \operatorname{curl} H\|_{\frac{2}{1-\varepsilon}}^{\frac{1-\varepsilon}{m(1-2\varepsilon)}} \\
& \leq C [Z(t) + m^{\frac{1}{2}} (\frac{m}{\varepsilon})^{\frac{1-s}{\beta-1}} \varphi(t)^{1-s} Z(t)^s] Z(t)^{2-\frac{1-\varepsilon}{m(1-2\varepsilon)}} \varphi(t)^{\frac{1-\varepsilon}{m(1-2\varepsilon)}} \\
& \leq C [m^{\frac{1}{2}} (\frac{m}{\varepsilon})^{\frac{1-s}{\beta-1}} \varphi(t)^{1-s+\frac{1-\varepsilon}{m(1-2\varepsilon)}} Z(t)^{2+s-\frac{1-\varepsilon}{m(1-2\varepsilon)}} \\
& \quad + Z(t)^{3-\frac{1-\varepsilon}{m(1-2\varepsilon)}} \varphi(t)^{\frac{1-\varepsilon}{m(1-2\varepsilon)}}] \\
& \leq \delta \varphi^2(t) + C_\delta [(1 + Z^2(t))^2 + (m^{\frac{1}{2}} (\frac{m}{\varepsilon})^{\frac{1-s}{\beta-1}} Z^s(t))^{\frac{2}{1+s}} \\
& \quad + (m^{\frac{m}{2(m-1)}} (\frac{m}{\varepsilon})^{\frac{m(1-s)}{(\beta-1)(m-1)}} Z(t)^{2+\frac{ms-1}{m-1}})^{\frac{2(m-1)}{m(1+s)-2}} \\
& \quad + (m^{\frac{1}{2}} (\frac{m}{\varepsilon})^{\frac{1-s}{\beta-1}} Z(t)^{2+s-\frac{1-\varepsilon}{m(1-2\varepsilon)}})^{\frac{2}{1+s-\frac{1-\varepsilon}{m(1-2\varepsilon)}}}] \\
& \leq \delta \varphi^2(t) + C_\delta [(1 + Z^2(t))^2 + m(\frac{m}{\varepsilon})^{\frac{2}{\beta-1}} (1 + Z^2(t))^{2+\frac{1-ms+(2ms-1)\varepsilon}{m(1+s)(1-2\varepsilon)-1+\varepsilon}}].
\end{aligned} \tag{3.46}$$

Substituting (3.41), (3.45) and (3.46) into (3.36) and choosing δ sufficiently small gives that

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} Z^2(t) + \frac{1}{2} \varphi^2(t) & \leq C [(1 + Z^2(t))^2 + (Z^2(t) + 1)^{2+\frac{\varepsilon}{1-3\varepsilon}} \varepsilon^{\frac{2(1-\varepsilon)}{(1-\beta)(1-3\varepsilon)}} \\
& \quad + m(\frac{m}{\varepsilon})^{\frac{2}{\beta-1}} (1 + Z^2(t))^{2+\frac{1-ms}{m(1+s)-2}} + m(\frac{m}{\varepsilon})^{\frac{2}{\beta-1}} (1 + Z^2(t)) \\
& \quad + m(\frac{m}{\varepsilon})^{\frac{2}{\beta-1}} (1 + Z^2(t))^{2+\frac{1-ms+(2ms-1)\varepsilon}{m(1+s)(1-2\varepsilon)-1+\varepsilon}}].
\end{aligned} \tag{3.47}$$

Notice that

$$\lim_{m \rightarrow +\infty} [2^m (1 - ms)] = 2,$$

hence $1 - ms \sim 2\varepsilon$ as $m \rightarrow +\infty$. For m sufficiently large enough, one can show that

$$\frac{1 - ms}{m(1 + s) - 2} \leq 4\varepsilon,$$

and

$$\frac{1 - ms + (2ms - 1)\varepsilon}{m(1 + s)(1 - 2\varepsilon) - 1 + \varepsilon} \leq 4\varepsilon.$$

Therefore, from (3.47), we obtain the following inequality

$$\frac{1}{2} \frac{d}{dt} Z^2(t) + \frac{1}{2} \varphi^2(t) \leq C m (\frac{m}{\varepsilon})^{\frac{2}{\beta-1}} (1 + Z^2(t))^{2+4\varepsilon}. \tag{3.48}$$

On the other hand, from

$$Z^2(t) = \int \mu \omega + (\operatorname{curl} H)^2 + \frac{(F + \frac{1}{2}|H|^2)^2}{2\mu + \lambda(\rho)} dx$$

and the estimate for density and the elementary estimates for the velocity and magnetic field, it is easy to show that $Z^2(t) \in L^1(0, T)$. Thus, from (3.48), we conclude

$$\frac{1}{(1 + Z^2(t))^{4\varepsilon}} - \frac{1}{(1 + Z^2(0))^{4\varepsilon}} + Cm\varepsilon\left(\frac{m}{\varepsilon}\right)^{\frac{2}{\beta-1}} \geq 0.$$

Take $M > 2$ so as to satisfy the inequality

$$CM\varepsilon\left(\frac{M}{\varepsilon}\right)^{\frac{2}{\beta-1}} \leq \frac{1}{2(1 + Z^2(0))^{4\varepsilon}},$$

that is,

$$CM^{1+\frac{2}{\beta-1}}2^{-m(1-\frac{2}{\beta-1})} \leq \frac{1}{2(1 + Z^2(0))^{4\varepsilon}}, \quad (3.49)$$

then

$$\frac{1}{(1 + Z^2(t))^{4\varepsilon}} \geq \frac{1}{(1 + Z^2(0))^{4\varepsilon}}. \quad (3.50)$$

Since

$$\begin{aligned} Z^2(0) &= \int \mu\omega_0 + (\operatorname{curl} H_0)^2 + \frac{(F_0 + \frac{1}{2}|H_0|^2)^2}{2\mu + \lambda(\rho_0)} dx \\ &\leq C[\|u_0\|_{H^2}^2 + \|H_0\|_{H^2}^2 + \|\rho_0\|_{H^3}^\beta \|u_0\|_{H^2}^2 + \|\rho_0\|_{H^3}^{2\gamma}] \\ &\leq C, \end{aligned}$$

if $1 - \frac{2}{\beta-1} > 0$, that is $\beta > 3$, then we can take sufficiently large $M > 2$ to ensure the condition (3.49). From (3.50), it holds that

$$Z^2(t) \leq 2^{2^{m-2}}(1 + Z^2(0)) - 1 \leq C$$

and

$$\int_0^T \varphi(t) dt \leq C.$$

Thus we complete the proof of Lemma 3.5.

Step 4. Second-order estimates for the velocity and the magnetic field

Lemma 3.6. *There exists a positive constant C independent of δ , such that*

$$\begin{aligned} &\sup_{t \in [0, T]} \int \rho(B^2 + L^2) + \nu |\nabla \operatorname{curl} H|^2 + |\nabla \rho|^2 dx + \int_0^T \int \mu(B_{x_1} - L_{x_2})^2 \\ &+ (2\mu + \lambda(\rho))(B_{x_2} + L_{x_1})^2 dx dt \leq C. \end{aligned}$$

Proof. Multiplying the first equation of the system (2.4) by B , the second by L , and integrating their sum in the space variable over \mathbb{T}^2 , then using the continuity

equation, we get

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int \rho(B^2 + L^2) dx + \int \mu(B_{x_1} - L_{x_2})^2 + (2\mu + \lambda(\rho))(B_{x_2} + L_{x_1})^2 dx \\
&= \int \rho(B^2 + L^2) \operatorname{div} u dx + \int \mu \left[\left(\frac{1}{2\rho} |H|_{x_2}^2 \right)_{x_1} - \left(\frac{1}{2\rho} |H|_{x_1}^2 \right)_{x_2} \right] (B_{x_1} - L_{x_2}) dx \\
&- \int \rho(2\mu + \lambda(\rho)) \left[\left(F + \frac{1}{2} |H|^2 \right) \left(\frac{1}{2\mu + \lambda(\rho)} \right)' + \left(\frac{p}{2\mu + \lambda(\rho)} \right)' \right] \operatorname{div} u (B_{x_2} + L_{x_1}) dx \\
&- \int B(u_{x_2} \cdot \nabla(F + \frac{1}{2} |H|^2) + \mu u_{x_1} \nabla \omega) + L(u_{x_1} \cdot \nabla(F + \frac{1}{2} |H|^2) - \mu u_{x_2} \nabla \omega) dx \\
&+ \int \mu [\omega \operatorname{div} u + \left(\frac{1}{\rho} H \cdot \nabla H_1 \right)_{x_2} - \left(\frac{1}{\rho} H \cdot \nabla H_1 \right)_{x_1}] (B_{x_1} - L_{x_2}) dx \\
&+ \int (2\mu + \lambda(\rho)) [(u_{1x_1})^2 + 2u_{1x_2} u_{2x_1} + (u_{2x_2})^2] (B_{x_2} + L_{x_1}) dx \\
&- \int \frac{2\mu + \lambda(\rho)}{\rho} [(H_{1x_1})^2 + 2H_{1x_2} H_{2x_1} + (H_{2x_2})^2] (B_{x_2} + L_{x_1}) dx \\
&- \int (2\mu + \lambda(\rho)) \left[\left(\frac{1}{\rho} \right)_{x_1} H \cdot \nabla H_1 + \left(\frac{1}{\rho} \right)_{x_2} H \cdot \nabla H_2 \right] (B_{x_2} + L_{x_1}) dx \\
&+ \int (2\mu + \lambda(\rho)) (B_{x_2} + L_{x_1}) \left[\left(\frac{1}{2\rho} |H|_{x_2}^2 \right)_{x_2} + \left(\frac{1}{2\rho} |H|_{x_1}^2 \right)_{x_1} \right].
\end{aligned} \tag{3.51}$$

Applying the operator $\nabla \operatorname{curl}$ to the magnetic equation, multiplying it by $\nabla \operatorname{curl} H$, and then integrating over \mathbb{T}^2 , we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\nabla \operatorname{curl} H\|_2^2 + \nu \|\nabla^2 \operatorname{curl} H\|_2^2 \\
&= - \int \nabla \operatorname{curl} u \cdot \nabla H \cdot \nabla \operatorname{curl} H + \operatorname{curl} u \cdot \nabla \nabla H \cdot \nabla \operatorname{curl} H \\
&+ \nabla u \cdot \nabla \operatorname{curl} H \cdot \nabla \operatorname{curl} H + u \cdot \nabla \nabla \operatorname{curl} H \cdot \nabla \operatorname{curl} H \\
&- \nabla \operatorname{curl} H \cdot \nabla u \cdot \nabla \operatorname{curl} H - \operatorname{curl} H \cdot \nabla \nabla u \cdot \nabla \operatorname{curl} H \\
&- \nabla H \cdot \nabla \operatorname{curl} u \cdot \nabla \operatorname{curl} H - H \cdot \nabla \nabla \operatorname{curl} u \cdot \nabla \operatorname{curl} H \\
&+ \operatorname{div} u \nabla \operatorname{curl} H \cdot \nabla \operatorname{curl} H + \operatorname{curl} H \cdot \nabla \operatorname{div} u \cdot \nabla \operatorname{curl} H \\
&+ \nabla H \operatorname{curl} \operatorname{div} u \cdot \nabla \operatorname{curl} H + H \cdot \nabla \operatorname{curl} \operatorname{div} u \cdot \nabla \operatorname{curl} H dx.
\end{aligned} \tag{3.52}$$

Applying ∇ to the mass equation, multiplying by $\nabla \rho$, then integrating the resulted equation by parts, one arrives at

$$\frac{1}{2} \frac{d}{dt} \|\nabla \rho\|_2^2 = - \int \nabla u |\nabla \rho|^2 dx - \frac{1}{2} \int \operatorname{div} u |\nabla \rho|^2 dx - \int \rho \nabla \rho \cdot \nabla \operatorname{div} u dx,$$

which together with (3.51)-(3.52) gives that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int \rho(B^2 + L^2) + |\nabla \operatorname{curl} H|^2 + |\nabla \rho|^2 dx \\
& + \int \mu(B_{x_1} - L_{x_2})^2 + (2\mu + \lambda(\rho))(B_{x_2} + L_{x_1})^2 + \nu |\nabla^2 \operatorname{curl} H|^2 dx \\
& = \int \rho(B^2 + L^2) \operatorname{div} u dx + \int \mu \left[\left(\frac{1}{2\rho} |H|_{x_2}^2 \right)_{x_1} - \left(\frac{1}{2\rho} |H|_{x_1}^2 \right)_{x_2} \right] (B_{x_1} - L_{x_2}) dx \\
& - \int \rho(2\mu + \lambda(\rho)) \left[\left(F + \frac{1}{2} |H|^2 \right) \left(\frac{1}{2\mu + \lambda(\rho)} \right)' + \left(\frac{p}{2\mu + \lambda(\rho)} \right)' \right] \operatorname{div} u (B_{x_2} + L_{x_2}) dx \\
& - \int B(u_{x_2} \cdot \nabla(F + \frac{1}{2} |H|^2) + \mu u_{x_1} \nabla \omega) + L(u_{x_1} \cdot \nabla(F + \frac{1}{2} |H|^2) - \mu u_{x_2} \nabla \omega) dx \\
& + \int \mu [\omega \operatorname{div} u + \left(\frac{1}{\rho} H \cdot \nabla H_1 \right)_{x_2} - \left(\frac{1}{\rho} H \cdot \nabla H_1 \right)_{x_1}] (B_{x_1} - L_{x_2}) dx \\
& + \int (2\mu + \lambda(\rho)) [(u_{1x_1})^2 + 2u_{1x_2} u_{2x_1} + (u_{2x_2})^2] (B_{x_2} + L_{x_1}) dx \\
& - \int \frac{2\mu + \lambda(\rho)}{\rho} [(H_{1x_1})^2 + 2H_{1x_2} H_{2x_1} + (H_{2x_2})^2] (B_{x_2} + L_{x_1}) dx \\
& - \int (2\mu + \lambda(\rho)) \left[\left(\frac{1}{\rho} \right)_{x_1} H \cdot \nabla H_1 + \left(\frac{1}{\rho} \right)_{x_2} H \cdot \nabla H_2 \right] (B_{x_2} + L_{x_2}) dx \\
& + \int (2\mu + \lambda(\rho)) (B_{x_2} + L_{x_1}) \left[\left(\frac{1}{2\rho} |H|_{x_2}^2 \right)_{x_2} + \left(\frac{1}{2\rho} |H|_{x_1}^2 \right)_{x_1} \right] \\
& - \int \nabla \operatorname{curl} u \cdot \nabla H \cdot \nabla \operatorname{curl} H + \operatorname{curl} u \cdot \nabla \nabla H \cdot \nabla \operatorname{curl} H + \nabla u \cdot \nabla \operatorname{curl} H \cdot \nabla \operatorname{curl} H \\
& + u \cdot \nabla \nabla \operatorname{curl} H \cdot \nabla \operatorname{curl} H - \nabla \operatorname{curl} H \cdot \nabla u \cdot \nabla \operatorname{curl} H - \operatorname{curl} H \cdot \nabla \nabla u \cdot \nabla \operatorname{curl} H \\
& - \nabla H \cdot \nabla \operatorname{curl} u \cdot \nabla \operatorname{curl} H - H \cdot \nabla \nabla \operatorname{curl} u \cdot \nabla \operatorname{curl} H + \operatorname{div} u \nabla \operatorname{curl} H \cdot \nabla \operatorname{curl} H \\
& + \operatorname{curl} H \cdot \nabla \operatorname{div} u \cdot \nabla \operatorname{curl} H + \nabla H \operatorname{curl} \operatorname{div} u \cdot \nabla \operatorname{curl} H + H \cdot \nabla \operatorname{curl} \operatorname{div} u \cdot \nabla \operatorname{curl} H dx \\
& - \int \nabla u |\nabla \rho|^2 dx - \frac{1}{2} \int \operatorname{div} u |\nabla \rho|^2 dx - \int \rho \nabla \rho \cdot \nabla \operatorname{div} u dx.
\end{aligned} \tag{3.53}$$

Put

$$Y(t) = \left(\int \rho(B^2 + L^2) + |\nabla \operatorname{curl} H|^2 + |\nabla \rho|^2 dx \right)^{\frac{1}{2}} \tag{3.54}$$

and

$$\psi(t) = \left(\int \mu(B_{x_1} - L_{x_2})^2 + (2\mu + \lambda(\rho))(B_{x_2} + L_{x_1})^2 + \nu |\nabla^2 \operatorname{curl} H|^2 dx \right)^{\frac{1}{2}}. \tag{3.55}$$

Note that

$$\begin{aligned}
\int (|\nabla B|^2 + |\nabla L|^2) dx &= \int (B_{x_1}^2 + B_{x_2}^2 + L_{x_1}^2 + L_{x_2}^2) dx \\
&= \int [(B_{x_1} - L_{x_2})^2 + (B_{x_2} + L_{x_1})^2] dx \\
&\leq \frac{1}{\mu} \psi^2(t).
\end{aligned}$$

Thus it holds that

$$\|\nabla(B, L)\|_2(t) \leq C\psi(t), \quad \forall t \in [0, T]. \quad (3.56)$$

Then it follows from the elliptic system

$$\mu\omega_{x_1} + (F + \frac{1}{2}|H|^2)_{x_2} = \rho B,$$

$$-\mu\omega_{x_2} + (F + \frac{1}{2}|H|^2)_{x_1} = \rho L,$$

that

$$\|\nabla(F + \frac{1}{2}|H|^2, \omega)\|_p \leq C\|\rho(B, L)\|_p, \quad \forall 1 < p < +\infty. \quad (3.57)$$

Furthermore, since $\int \mu\omega_{x_1} + (F + \frac{1}{2}|H|^2)_{x_2} dx = 0$, by the mean value theorem, there exists a point $x_* \in \mathbb{T}^2$, such that $(\omega_{x_1} + (F + \frac{1}{2}|H|^2)_{x_2})(x_*, t) = 0$, and so $B(x_*, t) = 0$. Similarly, there exists a point x_*^1 , such that $L(x_*^1, t) = 0$. Therefore, by the Poincare inequality, it holds that

$$\|(B, L)\|_p \leq C\|\nabla(B, L)\|_2, \quad \forall 1 < p < +\infty, \quad (3.58)$$

where C depend on p .

Now we estimate the right-hand side of (3.53) term by term. From the Hölder inequality, (3.58) and Lemma 3.4, it holds that

$$\begin{aligned} |\int \rho(B^2 + L^2) \operatorname{div} u dx| &= |\int \rho(B^2 + L^2) \frac{F + \frac{1}{2}|H|^2 + p(\rho)}{2\mu + \lambda(\rho)} dx| \\ &\leq \|\sqrt{\rho}(B, L)\|_2 \|(B, L)\|_4 \|\frac{\sqrt{\rho}(F + \frac{1}{2}|H|^2 + p(\rho))}{2\mu + \lambda(\rho)}\|_4 \\ &\leq CY(t)\psi(t)(1 + \|F + \frac{1}{2}|H|^2\|_4). \end{aligned} \quad (3.59)$$

Observe that

$$\|(F + \frac{1}{2}|H|^2, \omega)\|_4 \leq C\|\nabla(F + \frac{1}{2}|H|^2, \omega)\|_2 \leq CY(t). \quad (3.60)$$

Then we can write (3.59) as

$$\begin{aligned} \int \rho(B^2 + L^2) \operatorname{div} u dx &= \int \rho(B^2 + L^2) \frac{F + \frac{1}{2}|H|^2 + p(\rho)}{2\mu + \lambda(\rho)} dx \\ &\leq CY(t)\psi(t)(1 + Y(t)) \\ &\leq \delta\psi^2(t) + C_\delta(Y(t) + 1)^4. \end{aligned} \quad (3.61)$$

Direct estimates give

$$\begin{aligned}
& - \int \rho(2\mu + \lambda(\rho))[(F + \frac{1}{2}|H|^2)(\frac{1}{2\mu + \lambda(\rho)})' + (\frac{p}{2\mu + \lambda(\rho)})'] \operatorname{div} u(B_{x_2} + L_{x_2}) dx \\
& \leq \delta \int (2\mu + \lambda(\rho))(B_{x_2} + L_{x_2})^2 dx + C_\delta \int \rho^2(2\mu + \lambda(\rho))[(F + \frac{1}{2}|H|^2)(\frac{1}{2\mu + \lambda(\rho)})' \\
& \quad + (\frac{p}{2\mu + \lambda(\rho)})']^2 (\operatorname{div} u)^2 dx \\
& \leq \delta \psi^2(t) + C_\delta \int \rho^2[(F + \frac{1}{2}|H|^2)(\frac{1}{2\mu + \lambda(\rho)})' + (\frac{p}{2\mu + \lambda(\rho)})']^2 \frac{|F + \frac{1}{2}|H|^2|^2 + p^2(\rho)}{2\mu + \lambda(\rho)} dx \\
& \leq \delta \psi^2(t) + C_\delta(1 + \|F + \frac{1}{2}|H|^2\|_4^4) \\
& \leq \delta \psi^2(t) + C_\delta(Y(t) + 1)^4,
\end{aligned} \tag{3.62}$$

$$\begin{aligned}
& - \int B(u_{x_2} \cdot \nabla(F + \frac{1}{2}|H|^2) + \mu u_{x_1} \nabla \omega) + L(u_{x_1} \cdot \nabla(F + \frac{1}{2}|H|^2) - \mu u_{x_2} \nabla \omega) dx \\
& \leq C \int |(B, L)| |\nabla u| |\nabla(F + \frac{1}{2}|H|^2, \omega)| dx \leq C \|(B, L)\|_8 \|\nabla u\|_2 \|\nabla(F + \frac{1}{2}|H|^2, \omega)\|_{\frac{8}{3}} \\
& \leq C \|\nabla(B, L)\|_2 \|\rho(B, L)\|_{\frac{8}{3}} \leq CY^{\frac{3}{8}}(t) \psi^{\frac{13}{8}} \leq \delta \psi^2(t) + C_\delta Y^2(t),
\end{aligned} \tag{3.63}$$

$$\begin{aligned}
& + \int \mu[\omega \operatorname{div} u + (\frac{1}{\rho} H \cdot \nabla H_1)_{x_2} - (\frac{1}{\rho} H \cdot \nabla H_1)_{x_1}](B_{x_1} - L_{x_2}) dx \\
& \leq \mu \left(\int (B_{x_1} - L_{x_2})^2 dx \right)^{\frac{1}{2}} \left(\int \omega^2 (\operatorname{div} u)^2 + [(\frac{1}{\rho} H \cdot \nabla H_1)_{x_2} - (\frac{1}{\rho} H \cdot \nabla H_1)_{x_1}]^2 dx \right)^{\frac{1}{2}} \\
& \leq \delta \psi^2(t) + C_\delta \|\omega\|_4^2 \left\| \frac{F + \frac{1}{2}|H|^2 + p(\rho)}{2\mu + \lambda(\rho)} \right\|_4^2 + C_\delta Y^2(t) \\
& \leq \delta \psi^2(t) + C_\delta \|\omega\|_4^2 (1 + \|F + \frac{1}{2}|H|^2\|_4^2) + C_\delta Y^2(t) \\
& \leq \delta \psi^2(t) + C_\delta(Y(t) + 1)^4,
\end{aligned} \tag{3.64}$$

$$\begin{aligned}
& \int (2\mu + \lambda(\rho))[(u_{1x_1})^2 + 2u_{1x_2}u_{2x_1} + (u_{2x_2})^2](B_{x_2} + L_{x_1})dx \\
& - \int \frac{2\mu + \lambda(\rho)}{\rho}[(H_{1x_1})^2 + 2H_{1x_2}H_{2x_1} + (H_{2x_2})^2](B_{x_2} + L_{x_1})dx \\
& \leq \left(\int (2\mu + \lambda(\rho))(B_{x_2} + L_{x_1})^2 dx \right)^{\frac{1}{2}} \left(\int (2\mu + \lambda(\rho))[(u_{1x_1})^2 + 2u_{1x_2}u_{2x_1} \right. \\
& \quad \left. + (u_{2x_2})^2 + (H_{1x_1})^2 + 2H_{1x_2}H_{2x_1} + (H_{2x_2})^2] dx \right)^{\frac{1}{2}} \\
& \leq \delta\psi^2(t) + C_\delta \int (2\mu + \lambda(\rho))[(u_{1x_1})^2 + 2u_{1x_2}u_{2x_1} + (u_{2x_2})^2 + (H_{1x_1})^2 \\
& \quad + 2H_{1x_2}H_{2x_1} + (H_{2x_2})^2] dx \\
& \leq \delta\psi^2(t) + C_\delta \|2\mu + \lambda(\rho)\|_2 (\|\nabla u\|_8^4 + \|\nabla H\|_8^4) \\
& \leq \delta\psi^2(t) + C_\delta (\|\operatorname{div} u\|_8^4 + \|\omega\|_8^4 + \|\operatorname{curl} H\|_8^4) \\
& \leq \delta\psi^2(t) + C_\delta (1 + \|(F + \frac{1}{2}|H|^2, \omega)\|_8^4 + \|\operatorname{curl} H\|_8^4) \\
& \leq \delta\psi^2(t) + C_\delta (1 + \|\nabla(F + \frac{1}{2}|H|^2, \omega, \operatorname{curl} H)\|_{\frac{8}{5}}^4) \\
& \leq \delta\psi^2(t) + C_\delta (Y(t) + 1)^4,
\end{aligned} \tag{3.65}$$

where we have used the fact that

$$\|\nabla u\|_2 \leq C(\|\operatorname{div} u\|_2 + \|\omega\|_2) \leq C(\| \frac{F + \frac{1}{2}|H|^2 + p(\rho)}{2\mu + \lambda(\rho)} \|_2 + \|\omega\|_2) \leq C,$$

$$\begin{aligned}
\|\rho(B, L)\|_{\frac{8}{3}} &= \left(\int \sqrt{\rho} |(B, L)| |(B, L)|^{\frac{5}{3}} \rho^{\frac{13}{6}} dx \right)^{\frac{3}{8}} \leq \|\sqrt{\rho}(B, L)\|_2^{\frac{3}{8}} \|(B, L)\|_{\frac{5}{4}}^{\frac{5}{8}} \|\rho\|_{\frac{16}{26}}^{\frac{13}{8}} \\
&\leq CY^{\frac{3}{8}}(t) \|\nabla(B, L)\|_{\frac{5}{2}}^{\frac{5}{8}}.
\end{aligned}$$

After a tedious calculation,

$$\begin{aligned}
& - \int (2\mu + \lambda(\rho)) \left[\left(\frac{1}{\rho} \right)_{x_1} H \cdot \nabla H_1 + \left(\frac{1}{\rho} \right)_{x_2} H \cdot \nabla H_2 \right] (B_{x_2} + L_{x_2}) dx \\
& + \int (2\mu + \lambda(\rho))(B_{x_2} + L_{x_1}) \left[\left(\frac{1}{2\rho} |H|_{x_2}^2 \right)_{x_2} + \left(\frac{1}{2\rho} |H|_{x_1}^2 \right)_{x_1} \right] \\
& \int \mu \left[\left(\frac{1}{2\rho} |H|_{x_2}^2 \right)_{x_1} - \left(\frac{1}{2\rho} |H|_{x_1}^2 \right)_{x_2} \right] (B_{x_1} - L_{x_2}) dx \\
& \leq \delta\psi^2(t) + C_\delta (Y(t) + 1)^4,
\end{aligned} \tag{3.66}$$

$$\begin{aligned}
& - \int \nabla \operatorname{curl} u \cdot \nabla H \cdot \nabla \operatorname{curl} H + \operatorname{curl} u \cdot \nabla \nabla H \cdot \nabla \operatorname{curl} H + \nabla u \cdot \nabla \operatorname{curl} H \cdot \nabla \operatorname{curl} H \\
& + u \cdot \nabla \nabla \operatorname{curl} H \cdot \nabla \operatorname{curl} H - \nabla \operatorname{curl} H \cdot \nabla u \cdot \nabla \operatorname{curl} H - \operatorname{curl} H \cdot \nabla \nabla u \cdot \nabla \operatorname{curl} H \\
& - \nabla H \cdot \nabla \operatorname{curl} u \cdot \nabla \operatorname{curl} H - H \cdot \nabla \nabla \operatorname{curl} u \cdot \nabla \operatorname{curl} H + \operatorname{div} u \nabla \operatorname{curl} H \cdot \nabla \operatorname{curl} H \\
& + \operatorname{curl} H \cdot \nabla \operatorname{div} u \cdot \nabla \operatorname{curl} H + \nabla H \operatorname{curl} \operatorname{div} u \cdot \nabla \operatorname{curl} H + H \cdot \nabla \operatorname{curl} \operatorname{div} u \cdot \nabla \operatorname{curl} H dx \\
& \leq \delta\psi^2(t) + C(1 + Y(t))^4,
\end{aligned} \tag{3.67}$$

$$-\int \nabla u |\nabla \rho|^2 dx - \frac{1}{2} \int \operatorname{div} u |\nabla \rho|^2 dx - \int \rho \nabla \rho \cdot \nabla \operatorname{div} u dx \leq \delta \psi^2(t) + C(1 + Y(t))^4. \quad (3.68)$$

Incorporating (3.61)-(3.68) with (3.53) leads to

$$\frac{1}{2} \frac{d}{dt} Y^2(t) + \psi^2(t) \leq 8\delta \psi^2(t) + C_\delta(1 + Y^2(t))^2.$$

Choosing $8\delta = \frac{1}{2}$, from Lemma 3.5, we get $Y^2(t) \in L^1(0, T)$, and so using Gronwall's inequality gives that

$$Y^2(t) + \int_0^T \psi^2(t) dt \leq Y^2(0) + C. \quad (3.69)$$

From the system 1.1, it holds that

$$\begin{aligned} \mathcal{L}_{\rho_0} u_0 &= \mu \Delta u_0 + \nabla((\mu + \lambda(\rho_0)) \operatorname{div} u_0) = \mu \Delta u_0 + \nabla(F_0 - \mu \operatorname{div} u_0 + p(\rho_0) + \frac{1}{2}|H_0|) \\ &= [\mu \nabla(\operatorname{div} u_0) - \mu \nabla \times (\nabla \times u_0)] + \nabla(F_0 - \mu \operatorname{div} u_0 + p(\rho_0) + \frac{1}{2}|H_0|) \end{aligned}$$

with

$$F_0 = (2\mu + \lambda(\rho_0)) \operatorname{div} u_0 - p(\rho_0) - \frac{1}{2}|H_0|,$$

similarly one can define ω_0, B_0, L_0 . Thus

$$\begin{aligned} \mathcal{L}_{\rho_0} u_0 - \nabla p_0 &= \nabla(F_0 + \frac{1}{2}|H_0|) - \mu \nabla \times (\nabla \times u_0) \\ &= \nabla(F_0 + \frac{1}{2}|H_0|) - \mu(\partial_{x_2} \omega_0, -\partial_{x_1} \omega_0)^t \\ &= ((F_0 + \frac{1}{2}|H_0|)_{x_1} - \mu \partial_{x_2} \omega_0, (F_0 + \frac{1}{2}|H_0|)_{x_2} + \mu \partial_{x_1} \omega_0) \\ &= \rho_0(L_0, B_0)^t. \end{aligned}$$

Hence, there exists $g \in L^2(\mathbb{T}^2)$ such that

$$\sqrt{\rho_0} g = \rho_0(L_0, B_0)^t,$$

which gives that

$$Y^2(0) = \|\sqrt{\rho_0}(L_0, B_0)\|_2^2 = \|\frac{\sqrt{\rho_0} g}{\sqrt{\rho_0}}\|_2^2 \leq C.$$

Therefore, from (3.69), it holds that

$$Y^2(t) + \int_0^T \psi^2(t) dt \leq C.$$

□

The Lemma 3.6 is proved.

Step 5. Upper bound of the density

Lemma 3.7. *It holds that*

$$\int_0^T \|(F + \frac{1}{2}|H|^2, \omega)\|_{L^\infty}^3 dt \leq C.$$

Proof. From (3.57) with $p = 3$, it holds that

$$\begin{aligned}
\int_0^T \|\nabla(F + \frac{1}{2}|H|^2, \omega)\|_3^3 dt &\leq C \int_0^T \|\rho(B, L)\|_3^3 dt \\
&\leq C \int_0^T \int \rho^3 |(B, L)|^3 dx dt \\
&\leq C \int_0^T \int \rho^{\frac{1}{2}} |(B, L)| |(B, L)|^2 \rho^{\frac{5}{2}} dd t \\
&\leq C \int_0^T \|\rho^{\frac{1}{2}}(B, L)\|_2 \|(B, L)\|_8^2 \|\rho\|_{10}^{\frac{2}{5}} dt \\
&\leq C \int_0^T \|\nabla(B, L)\|_2^2 dt \\
&\leq C \int_0^T \psi^2(t) dt \leq C,
\end{aligned} \tag{3.70}$$

which together with Lemma 2.3 gives that

$$\int_0^T \|(F + \frac{1}{2}|H|^2, \omega)\|_\infty^3 \leq \int_0^T \|(F + \frac{1}{2}|H|^2, \omega)\|_{W^{1,3}}^3 dt \leq C. \tag{3.71}$$

□

Lemma 3.8. *It holds that*

$$\rho(x, t) \leq C, \quad \forall (x, t) \in \mathbb{T}^2 \times [0, T].$$

Proof. Using the continuity equation of (1.1), and the definition of $\theta(\rho)$, one gets

$$\theta(\rho)_t + u \cdot \nabla \rho + F + \frac{1}{2}|H|^2 + p(\rho) = 0. \tag{3.72}$$

Introducing the particle path $\vec{X}(x, t; \tau)$ through the point $(x, t) \in \mathbb{T} \times [0, T]$ defined by

$$\begin{cases} \frac{d\vec{X}(x, t; \tau)}{d\tau} = u(\vec{X}(x, t; \tau), \tau), \\ \vec{X}(x, t; \tau)|_{\tau=t} = x, \end{cases} \tag{3.73}$$

then we can show that

$$\frac{d}{d\tau} \theta(\rho)(\vec{X}(x, t; \tau), \tau) = -p(\rho)(\vec{X}(x, t; \tau), \tau) - F(\vec{X}(x, t; \tau), \tau) - \frac{1}{2}|H|^2(\vec{X}(x, t; \tau), \tau). \tag{3.74}$$

Integrating the above equality over $[0, t]$ implies that

$$\theta(\rho)(x, t) - \theta(\rho_0)(\vec{X}_0) = - \int_0^t (p(\rho) + F + \frac{1}{2}|H|^2)(\vec{X}(x, t; \tau), \tau) d\tau, \tag{3.75}$$

with $\vec{X}_0 = \vec{X}(x, t; \tau)|_{\tau=0}$.

From (3.75), it follows that

$$\begin{aligned}
2\mu \ln \frac{\rho(x, t)}{\rho_0(\vec{X}_0)} + \frac{1}{\beta} \rho^\beta(x, t) + \int_0^t p(\rho)(\vec{X}(x, t; \tau), \tau) d\tau \\
= \frac{1}{\beta} \rho_0^\beta(\vec{X}_0) - \int_0^t (F + \frac{1}{2}|H|^2)(\vec{X}(x, t; \tau), \tau) d\tau.
\end{aligned}$$

Thus

$$2\mu \ln \frac{\rho(x, t)}{\rho_0(\bar{X}_0)} \leq \frac{1}{\beta} \|\rho_0\|_\infty^\beta + \int_0^t \|(F + \frac{1}{2}|H|^2)(\tau, \cdot)\|_\infty d\tau \leq C,$$

which yields that

$$\rho(x, t) \leq C\rho_0(\bar{X}_0).$$

Hence,

$$\rho(x, t) \leq C, \quad \forall (x, t) \in \mathbb{T}^2 \times [0, T].$$

□

Lemma 3.9. *It holds that for any $1 < p < +\infty$,*

$$\begin{aligned} \int_0^T \|\operatorname{div} u\|_\infty^3 + \|\nabla(F + \frac{1}{2}|H|^2, \omega)\|_p^2 dt &\leq C, \\ \rho(x, t) &\geq m_1 > 0, \text{ with } m_1 \text{ a positive constant.} \end{aligned}$$

Proof. For any $1 < p < +\infty$,

$$\begin{aligned} \int_0^T \|\nabla(F + \frac{1}{2}|H|^2, \omega)\|_p^2 dt &\leq C \int_0^T \|\rho(B, L)\|_p^2 dt \leq C \int_0^T \|(B, L)\|_p^2 dt \\ &\leq C \int_0^T \|\nabla(B, L)\|_2^2 dt \leq C. \end{aligned}$$

On the other hand, it holds that

$$\int_0^T \|\operatorname{div} u\|_\infty^3 dt \leq C \int_0^T \|F + \frac{1}{2}|H|^2\|_\infty^3 + \|p(\rho)\|_\infty^3 dt \leq C.$$

From the continuity equation and the above estimate, it is quite easy to show that

$$\rho(x, t) \geq m_1 > 0.$$

Thus we complete the proof of Lemma 3.9. □

4. HIGHER ORDER ESTIMATES

In this section we derive some uniform estimates on their higher order estimates by virtue of the approximate solutions and basic estimates.

Lemma 4.1. *It holds that for any $1 \leq p < +\infty$,*

$$\sup_{t \in [0, T]} \|(\nabla \rho, \nabla p(\rho))(t, \cdot)\|_p + \int_0^T \|\nabla u\|_\infty^2 dt \leq C. \quad (4.1)$$

Proof. Applying the operator ∇ to the continuity equation of (1.1), then multiplying the resulted equation by $p|\nabla \rho|^{p-2} \nabla \rho$ with $p \geq 2$, integrating it in the space variable x over \mathbb{T}^2 implies that

$$\begin{aligned} \frac{d}{dt} \|\nabla \rho\|_p^p &= -(p-1) \int |\nabla \rho|^p \operatorname{div} u dx - p \int |\nabla \rho|^{p-2} \nabla \rho \cdot \nabla u \cdot \nabla \rho dx \\ &\quad - p \int \rho |\nabla \rho|^{p-2} \nabla \rho \cdot \nabla \operatorname{div} u dx \\ &\leq (p-1) \|\operatorname{div} u\|_\infty \|\nabla \rho\|_p^p + p \|\nabla u\|_\infty \|\nabla \rho\|_p^p + p \|\nabla \operatorname{div} u\|_p \|\nabla \rho\|_p^{p-1} \|\rho\|_\infty. \end{aligned}$$

This gives that

$$\begin{aligned}
\frac{d}{dt} \|\nabla \rho\|_p &\leq C[\|\nabla u\|_\infty \|\nabla \rho\|_p + \|\nabla \operatorname{div} u\|_p] \\
&\leq C[\|\nabla u\|_\infty \|\nabla \rho\|_p + \|\nabla(\frac{F + \frac{1}{2}|H|^2 + p(\rho)}{2\mu + \lambda(\rho)})\|_p] \\
&\leq C[(\|\nabla u\|_\infty + \|F + \frac{1}{2}|H|^2\|_\infty + 1)\|\nabla \rho\|_p + \|\nabla(F + \frac{1}{2}|H|^2)\|_p].
\end{aligned} \tag{4.2}$$

Since

$$\mathcal{L}_\rho u = \nabla p(\rho) + \rho(L, B)^t, \tag{4.3}$$

by elliptic estimates and (3.58), we show that for any $1 < p < +\infty$,

$$\begin{aligned}
\|\nabla^2 u\|_p &\leq C[\|\nabla p(\rho)\|_p + \|\rho(L, B)\|_p] \\
&\leq C[\|\nabla \rho\|_p + \|(L, B)\|_p] \\
&\leq C[\|\nabla \rho\|_p + \|\nabla(L, B)\|_2].
\end{aligned}$$

From the Beal-Kato-Majda type inequality, it follows that

$$\begin{aligned}
\|\nabla u\|_\infty &\leq C(\|\operatorname{div} u\|_\infty + \|\omega\|_\infty) \ln(e + \|\nabla^2 u\|_3) \\
&\leq C(\|\operatorname{div} u\|_\infty + \|\omega\|_\infty) \ln(e + \|\nabla \rho\|_3) \\
&\quad + C(\|\operatorname{div} u\|_\infty + \|\omega\|_\infty) \ln(e + \|\nabla(L, B)\|_2),
\end{aligned} \tag{4.4}$$

which together with (4.2) for $p = 3$, leads to

$$\begin{aligned}
\frac{d}{dt} \|\nabla \rho\|_3 &\leq C(\|\operatorname{div} u\|_\infty + \|\omega\|_\infty) \ln(e + \|\nabla \rho\|_3) \|\nabla \rho\|_3 \\
&\quad + C[(\|\operatorname{div} u\|_\infty + \|\omega\|_\infty) \ln(e + \|\nabla(L, B)\|_2) \\
&\quad + \|F + \frac{1}{2}|H|^2\|_\infty + 1] \|\nabla \rho\|_3 + C\|\nabla(F + \frac{1}{2}|H|^2)\|_p.
\end{aligned}$$

From (3.71), Lemma 3.9 and the Gronwall's inequality, we get

$$\sup_{t \in [0, T]} \|\nabla \rho\|_3 \leq C. \tag{4.5}$$

Incorporating Lemma 3.7, Lemma 3.9 and (4.4)-(4.5) yields that

$$\int_0^T \|\nabla u\|_\infty^2 dt \leq C. \tag{4.6}$$

Again using Lemma 3.7, Lemma 3.9 and (4.2), by Gronwall's inequality, one arrives at

$$\sup_{t \in [0, T]} \|\nabla \rho\|_p \leq C(\|\nabla \rho_0\|_p + 1), \quad \forall p \in [1, +\infty). \tag{4.7}$$

□

Lemma 4.2. *It holds that for any $1 \leq p < +\infty$,*

$$\begin{aligned}
&\sup_{t \in [0, T]} [\|u(t, \cdot)\|_\infty + \|\nabla u\|_p + \|(\rho_t, p_t)\|_p + \|(\rho_t, p_t)\|_{H^1} + \|(\rho, p(\rho), u)\|_{H^2}] \\
&+ \int_0^T \|u\|_{H^3}^2 dt \leq C.
\end{aligned}$$

Proof. From elliptic estimates and (4.3), it holds that

$$\begin{aligned} \sup_{t \in [0, T]} \|u\|_{H^2} &\leq C \sup_{t \in [0, T]} (\|\nabla p(\rho)\|_2 + \|\rho(L, B)\|_2) \\ &\leq C \sup_{t \in [0, T]} (\|\nabla p(\rho)\|_2 + \|\sqrt{\rho}(L, B)\|_2) \leq C. \end{aligned} \quad (4.8)$$

By virtue of Sobolev embedding theorem, one gets

$$\sup_{t \in [0, T]} |u(x, t)| \leq C, \quad \sup_{t \in [0, T]} \|\nabla u\|_p \leq C, \quad \forall 1 \leq p < +\infty. \quad (4.9)$$

From

$$\rho_t = -u \cdot \nabla \rho - \rho \operatorname{div} u$$

and

$$p_t = -u \cdot \nabla p - \rho p'(\rho) \operatorname{div} u, \quad (4.10)$$

together with the uniform upper bound of the density, Lemma 4.1 and (4.9), one can show that

$$\sup_{t \in [0, T]} \|(\rho_t, p_t)\|_p \leq C, \quad \forall p \in [1, +\infty). \quad (4.11)$$

Applying the operator ∇^2 to the continuity equation in (1.1) and (4.10), multiplying the resulted equation by $\nabla^2 \rho$ and $\nabla^2 p(\rho)$, integrating them over the torus \mathbb{T}^2 , we can prove that

$$\begin{aligned} \frac{d}{dt} \|\nabla^2 \rho\|_2^2 &\leq C[\|\nabla u\|_\infty \|\nabla^2 \rho\|_2^2 + \|\nabla \rho\|_3 \|\nabla^2 \rho\|_2 \|\nabla^2 u\|_6 + \|\rho\|_{L^\infty} \|\nabla^2 \rho\|_2 \|\nabla^3 u\|_2] \\ &\leq C[(\|\nabla u\|_\infty + 1) \|\nabla^2 \rho\|_2^2 + \|\nabla^3 u\|_2^2 + 1], \end{aligned} \quad (4.12)$$

$$\begin{aligned} \frac{d}{dt} \|\nabla^2 p(\rho)\|_2^2 &\leq C[\|\nabla^2 u\|_6 \|\nabla p\|_3 \|\nabla^2 p\|_2 + \|\nabla \rho\|_3 \|\nabla^2 p\|_2 \|\nabla^2 u\|_6 \|p'(\rho)\|_\infty \\ &\quad + \|\nabla u\|_\infty \|\nabla^2 p\|_2^2 + \|p''(\rho)\rho\|_{L^\infty} \|\nabla^2 u\|_6 \|\nabla^2 p\|_2 \|\nabla \rho\|_6^2 \\ &\quad + \|p'(\rho)\rho\|_{L^\infty} \|\nabla^3 u\|_2 \|\nabla^2 p\|_2 + \|p'(\rho)\|_{L^\infty} \|\nabla u\|_\infty \|\nabla^2 p\|_2 \|\nabla^2 \rho\|_2 \\ &\quad + \|p''(\rho)\|_{L^\infty} \|\nabla u\|_\infty \|\nabla^2 p\|_2 \|\nabla \rho\|_4^2 + \|p'''(\rho)\rho\|_{L^\infty} \|\nabla u\|_\infty \|\nabla^2 p\|_2 \|\nabla \rho\|_4^2 \\ &\quad + \|p''(\rho)\rho\|_{L^\infty} \|\nabla u\|_\infty \|\nabla^2 p\|_2 \|\nabla^2 \rho\|_2] \\ &\leq C[(\|\nabla u\|_\infty + 1) \|\nabla^2 p\|_2^2 + \|\nabla^3 u\|_2^2 + \|\nabla u\|_\infty \|\nabla^2 \rho\|_2^2]. \end{aligned} \quad (4.13)$$

From (4.3), it holds that

$$\mathcal{L}(\nabla u) = \nabla^2 p(\rho) + \nabla[\rho(L, B)] + \nabla(\nabla \lambda(\rho) \operatorname{div} u) := \phi.$$

Then the standard elliptic estimates imply that

$$\begin{aligned} \|u\|_{H^3} &\leq C[\|u\|_{H^1} + \|\phi\|_2] \\ &\leq C[\|u\|_{H^1} + \|\nabla^2 p\|_2 + \|\nabla \rho\|_4 \|(L, B)\|_4 + \|\rho\|_\infty \|\nabla(L, B)\|_2 \\ &\quad + \|\nabla^2 \rho\|_2 \|\nabla u\|_\infty + \|\nabla \rho\|_3 \|\nabla^2 u\|_6], \end{aligned}$$

and

$$\|\nabla^2 u\|_6 \leq C(\|\nabla p \rho\|_6 + \|\rho(L, B)\|_6) \leq C(1 + \|\nabla(L, B)\|_2).$$

Hence,

$$\|u\|_{H^3} \leq C[1 + \|\nabla^p\|_2 + \|\nabla(L, B)\|_2 + \|\nabla u\|_\infty \|\nabla^2 \rho\|_2],$$

which together with (4.12)-(4.13) gives that

$$\frac{d}{dt} \|(\nabla^2 \rho, \nabla^2 p(\rho))\|_2^2 \leq C[(\|\nabla u\|_\infty^2 + 1) \|(\nabla^2 \rho, \nabla^2 p(\rho))\|_2^2 + \|\nabla(L, B)\|_2 + 1].$$

Using the Gronwall's inequality, one can show that

$$\begin{aligned} \|(\nabla^2 \rho, \nabla^2 p(\rho))\|_2^2 &\leq \|(\nabla^2 \rho_0, \nabla^2 p_0)\|_2^2 + C \int_0^T (\|\nabla(L, B)\|_2 + 1) dt e^{\int_0^T (\|\nabla u\|_\infty^2 + 1) dt} \\ &\leq C, \end{aligned}$$

which gives that

$$\sup_{t \in [0, T]} (\|(\rho, p(\rho))\|_{H^2} + \|(\rho_t, p_t)\|_{H^1}) + \int_0^T \|u\|_{H^3}^2 dt \leq C.$$

The proof of Lemma 4.2 is completed. \square

Lemma 4.3. *It holds that*

$$\int_0^T \|H_t\|_2^2 dt \leq C,$$

$$\sup_{t \in [0, T]} \|\sqrt{\rho} u_t\|_2^2 + \int_0^T \|u_t\|_{H^1}^2 dt \leq C.$$

Proof. From the magnetic equation in (2.4), one gets

$$\begin{aligned} \|H_t\|_2 &\leq C[\|u \cdot \nabla H\|_2 + \|\Delta H\|_2 + \|H \cdot \nabla u\|_2 + \|H \operatorname{div} u\|_2] \\ &\leq C[\|u\|_\infty \|\nabla H\|_2 + \|\Delta H\|_2 + \|\nabla u\|_2 \|H\|_\infty], \end{aligned}$$

which yields that

$$\int_0^T \|H_t\|_2^2 dt \leq C.$$

The momentum equation in (1.1) can be rewritten as

$$\rho u_t + \rho u \cdot \nabla u + \nabla p(\rho) + \nabla \left(\frac{1}{2} |H|^2 \right) - H \cdot \nabla H = \mathcal{L}_\rho u := \mu \Delta u + \nabla((\mu + \lambda(\rho)) \operatorname{div} u). \quad (4.14)$$

Applying ∂_t to the above equation gives that

$$\begin{aligned} \rho u_{tt} + \rho u \cdot \nabla u_t + \nabla p(\rho)_t + \nabla \left(\frac{1}{2} |H_t|^2 \right) - H_t \cdot \nabla H - H \cdot \nabla H_t + \rho_t u_t \\ = \mu \Delta u_t + \nabla((\mu + \lambda(\rho)) \operatorname{div} u_t) - \rho_t u \cdot \nabla u - \rho u_t \cdot \nabla u + \nabla(\lambda(\rho)_t \operatorname{div} u). \end{aligned} \quad (4.15)$$

Multiplying the above equation by u_t and integrating the resulting equation over \mathbb{T}^2 gives that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int \rho |u_t|^2 dx + \int (\mu |\nabla u_t|^2 + (\mu + \lambda(\rho)) |\operatorname{div} u_t|^2) dx \\
&= - \int \nabla p(\rho)_t \cdot u_t dx - \int \nabla \left(\frac{1}{2} |H|_t^2 \right) \cdot u_t dx + \int H_t \cdot \nabla H \cdot u_t dx \\
&+ \int H \cdot \nabla H_t \cdot u_t dx - \int \rho_t |u_t|^2 dx - \int \rho_t u \cdot \nabla u \cdot u_t dx \\
&- \int \rho (u_t \cdot \nabla u) \cdot u_t dx + \int \nabla (\lambda(\rho)_t \operatorname{div} u) \cdot u_t dx.
\end{aligned} \tag{4.16}$$

The terms on the right-hand side of (4.16) can be estimated as

$$\begin{aligned}
& - \int \nabla p(\rho)_t \cdot u_t dx = \int p(\rho)_t \operatorname{div} u_t dx \leq \frac{\mu}{8} \|\operatorname{div} u_t\|_2^2 + C \|p(\rho)_t\|_2^2 \leq \frac{\mu}{8} \|\operatorname{div} u_t\|_2^2 + C, \\
& - \int \nabla \left(\frac{1}{2} |H|_t^2 \right) \cdot u_t dx = \int \frac{1}{2} |H|_t^2 \operatorname{div} u_t dx = \int H \cdot H_t \operatorname{div} u_t dx \leq \frac{\mu}{8} \|\operatorname{div} u_t\|_2^2 + C \|H_t\|_2^2, \\
& \int H_t \cdot \nabla H \cdot u_t dx = - \int H_t \cdot H \cdot \nabla u_t dx \leq \frac{\mu}{8} \|\nabla u_t\|_2^2 + C \|H_t\|_2^2, \\
& \int H \cdot \nabla H_t \cdot u_t dx = - \int H \cdot H_t \cdot \nabla u_t dx \leq \frac{\mu}{8} \|\nabla u_t\|_2^2 + C \|H_t\|_2^2, \\
& - \int \rho_t |u_t|^2 dx = \int \operatorname{div}(\rho u) |u_t|^2 dx = -2 \int \rho u \cdot \nabla u_t \cdot u_t dx \\
& \leq \|\nabla u_t\|_2 \|\sqrt{\rho} u_t\|_2 \|\sqrt{\rho}\|_\infty \|u\|_\infty \leq \frac{\mu}{8} \|\nabla u_t\|_2^2 + C \|\sqrt{\rho} u_t\|_2 \|\sqrt{\rho}\|_\infty^2 \|u\|_\infty^2 \\
& \leq \frac{\mu}{8} \|\nabla u_t\|_2^2 + C \|\sqrt{\rho} u_t\|_2, \\
& - \int \rho_t u \cdot \nabla u \cdot u_t dx = \int \operatorname{div}(\rho u) [(u \cdot \nabla u) \cdot u_t] dx = - \int \rho u \cdot \nabla [(u \cdot \nabla u) \cdot u_t] dx \\
& \leq \|\nabla u_t\|_2 \|\nabla u\|_2 \|\rho\|_\infty \|u\|_\infty^2 + (\|\nabla u\|_4^2 + \|u\|_\infty \|\nabla^2 u\|_2) \|\sqrt{\rho} u_t\|_2 \|\sqrt{\rho}\|_\infty \|u\|_\infty \\
& \leq \frac{\mu}{8} \|\nabla u_t\|_2^2 + C (\|\sqrt{\rho} u_t\|_2^2 + \|\nabla u\|_4^4 + \|(\nabla u, \nabla^2 u)\|_2^2) \\
& \leq \frac{\mu}{8} \|\nabla u_t\|_2^2 + C (\|\sqrt{\rho} u_t\|_2^2 + 1), \\
& - \int \rho (u_t \cdot \nabla u) \cdot u_t dx \leq \|\nabla u\|_\infty \|\sqrt{\rho} u_t\|_2^2, \\
& \int \nabla (\lambda(\rho)_t \operatorname{div} u) \cdot u_t dx = - \int \lambda(\rho)_t \operatorname{div} u \operatorname{div} u_t dx \\
& \leq \frac{\mu}{8} \|\operatorname{div} u_t\|_2^2 + C \|\lambda' \rho\|_\infty^2 \|\rho_t\|_{H^1}^2 \|\operatorname{div} u\|_4^2 \leq \frac{\mu}{8} \|\operatorname{div} u_t\|_2^2 + C,
\end{aligned}$$

which combining with (4.16) implies that

$$\begin{aligned} \|\sqrt{\rho}u_t\|_2^2 + \int_0^t \|\nabla u_t\|_2^2 d\tau &\leq \|\sqrt{\rho_0}u_t(0)\|_2^2 + \frac{\mu}{8} \|\operatorname{div} u_t\|_2^2 + C \int_0^t (\|\nabla u\|_\infty + 1) \|\sqrt{\rho}u_t\|_2^2 d\tau \\ &\quad + C \int_0^t \|H_t\|_2^2 d\tau + C. \end{aligned}$$

From the momentum equation of (1.1), we have

$$\begin{aligned} \|\sqrt{\rho_0}u_t(0)\|_2^2 &\leq \left\| \frac{\sqrt{\rho_0}}{\sqrt{\rho_0}} g \right\|_2^2 + \|\sqrt{\rho_0}\|_\infty^2 \|u_0\|_\infty^2 \|\nabla u_0\|_2^2 + \left\| \frac{1}{\sqrt{\rho_0}} \right\|_\infty^2 \|H_0\|_\infty^2 \|\nabla H_0\|_2^2 \\ &\quad + \|H_0\|_\infty^2 \|\nabla H_0\|_2^2 \leq C. \end{aligned}$$

Therefore, by the Gronwall's inequality, we get

$$\sup_{t \in [0, T]} \|\sqrt{\rho}u_t\|_2^2 + \int_0^T \|\nabla u_t\|_2^2 dt \leq C.$$

Note that

$$u_t = (L, B)^t - u \cdot \nabla u + \frac{1}{\rho} H \cdot \nabla H,$$

then for any $1 \leq p < +\infty$,

$$\begin{aligned} \int_0^T \|u_t\|_p^2 dt &\leq \int_0^T \|(L, B)\|_p^2 + \|u\|_\infty^2 \|\nabla u\|_p^2 + \left\| \frac{1}{\rho} \right\|_\infty^2 \|H\|_\infty^2 \|\nabla H\|_p^2 dt \\ &\leq \int_0^T \|\nabla(L, B)\|_2^2 + \|u\|_\infty^2 \|\nabla u\|_p^2 + \left\| \frac{1}{\rho} \right\|_\infty^2 \|H\|_\infty^2 \|\nabla H\|_p^2 dt \leq C. \end{aligned}$$

Thus, we arrive at

$$\int_0^T \|u_t\|_{H^1}^2 dt \leq C.$$

Hence the proof of Lemma 4.3 is finished. \square

Lemma 4.4. *It holds that*

$$\sup_{t \in [0, T]} \|(\rho_t, p(\rho)_t, \lambda(\rho)_t)\|_{H^1} + \int_0^T \|(\rho_{tt}, p(\rho)_{tt}, \lambda(\rho)_{tt})\|_2^2 dt \leq C.$$

Proof. Based on the continuity equation, it holds that

$$\rho_t = -u \cdot \nabla \rho - \rho \operatorname{div} u$$

and

$$\rho_{tt} = -u_t \cdot \nabla \rho - u \cdot \nabla \rho_t - \rho_t \operatorname{div} u - \rho \operatorname{div} u_t.$$

Clearly

$$\sup_{t \in [0, T]} \|\nabla \rho_t\|_2 \leq \sup_{t \in [0, T]} [\|\nabla \rho\|_4 \|\nabla u\|_4 + \|u\|_\infty \|\nabla^2 \rho\|_2 + \|\rho\|_\infty \|\nabla^2 u\|_2] \leq C, \quad (4.17)$$

$$\begin{aligned} \int_0^T \|\rho_{tt}\|_2^2 dt &\leq \int_0^T [\|u_t\|_4^2 \|\nabla \rho\|_4^2 + \|u\|_\infty^2 \|\nabla \rho_t\|_2^2 + \|\rho_t\|_4^2 \|\nabla u\|_4^2 + \|\rho\|_\infty^2 \|\nabla u_t\|_2^2] \\ &\leq C \int_0^T (\|u_t\|_{H^1}^2 + 1) dt \leq C. \end{aligned} \quad (4.18)$$

By a similar calculation, we also obtain

$$\sup_{t \in [0, T]} \|\nabla(p(\rho)_t, \lambda(\rho)_t)\|_2 + \int_0^T \|(p(\rho)_{tt}, \lambda(\rho)_{tt})\|_2^2 dt \leq C.$$

Thus we finish the proof of Lemma 4.4. \square

Lemma 4.5. *It holds that for any $0 < t \leq T$,*

$$\sup_{t \in [0, T]} \|H\|_{H^2}^2 + \int_0^T \|H\|_{H^3}^2 dt \leq C.$$

Proof. Applying $\partial_{x_i x_j}$, $i, j = 1, 2$, to the magnetic equation in (2.4), multiplying it by $\partial_{x_i x_j} H$, and then integrating the resulted equation in the space variable x , we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla^2 H\|_2^2 + \nu \|\nabla^3 H\|_2^2 &= - \int [\partial_{x_i x_j} u \cdot \nabla H + \partial_{x_i} u \cdot \nabla H_{x_j} + u \cdot \nabla H_{x_i x_j} \\ &\quad + \partial_{x_j} u \cdot \nabla H_{x_i} - \partial_{x_i x_j} H \cdot \nabla u - \partial_{x_i} H \cdot \nabla u_{x_j} - \partial_{x_j} H \cdot \nabla u_{x_i} - H \cdot \nabla u_{x_i x_j} \\ &\quad + \partial_{x_i x_j} H \operatorname{div} u + \partial_{x_i} H \operatorname{div} u_{x_j} + \partial_{x_j} H \operatorname{div} u_{x_i} + H \operatorname{div} u_{x_i x_j}] \cdot \partial_{x_i x_j} H dx \\ &\leq \frac{\nu}{2} \|\nabla^3 H\|_2^2 + C(\|\nabla u\|_\infty + 1) \|\nabla^2 H\|_2^2 + C \|\nabla^3 u\|_2^2 + C, \end{aligned}$$

that is,

$$\frac{1}{2} \frac{d}{dt} \|\nabla^2 H\|_2^2 + \nu \|\nabla^3 H\|_2^2 \leq C(\|\nabla u\|_\infty + 1) \|\nabla^2 H\|_2^2 + C \|\nabla^3 u\|_2^2 + C.$$

It follows from the Gronwall's inequality that

$$\begin{aligned} \sup_{t \in [0, T]} \|\nabla^2 H\|_2^2 + \int_0^T \|\nabla^3 H\|_2^2 dt &\leq (\|\nabla^2 H_0^\delta\|_2^2 + C \int_0^T (\|\nabla^3 u\|_2^2 + 1) dt) \\ &\quad \times e^{\int_0^T (\|\nabla u\|_\infty + 1) dt} \leq C. \end{aligned}$$

Therefore, together with Lemma 3.5, Lemma 4.1, Lemma 4.2 and elementary estimate (3.1), it holds that

$$\sup_{t \in [0, T]} \|\nabla^2 H\|_2^2 + \int_0^T \|H\|_{H^3}^2 dt \leq C.$$

\square

Lemma 4.6. *It holds that*

$$\begin{aligned} \sup_{t \in [0, T]} \|H_t\|_2^2 + \int_0^T \|H_t\|_{H^1}^2 dt &\leq C, \\ \sup_{t \in [0, T]} \|H\|_{H^2}^2 + \int_0^T \|\nabla^2 H\|_q^2 dt &\leq C, \\ \sup_{t \in [0, T]} [t\|u_t\|_{H^1}^2 + t\|u\|_{H^3}^2 + t\|(\rho_{tt}, p(\rho)_{tt}, \lambda(\rho)_{tt})\|_2^2 + \|(\rho, p(\rho))\|_{W^{2,q}}] \\ &\quad + \int_0^T t[\|\sqrt{\rho} u_{tt}\|_2^2 + \|u_t\|_{H^2}^2 + \|u\|_{H^4}^2] dt \leq C. \end{aligned} \tag{4.19}$$

Proof. From the magnetic equation in (2.4), we see that

$$H_{tt} + u_t \cdot \nabla H + u \cdot \nabla H_t - \nu \Delta H_t - H_t \cdot \nabla u - H \cdot \nabla u_t + H_t \operatorname{div} u + H \operatorname{div} u_t = 0.$$

Multiplying the above equality by H_t implies that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|H_t\|_2^2 + \nu \|\nabla H_t\|_2^2 &= - \int u_t \cdot \nabla H \cdot H_t dx - \int u \cdot \nabla H_t \cdot H_t dx \\ &\quad + \int H_t \cdot \nabla u \cdot H_t dx + \int H \cdot \nabla u_t \cdot H_t dx \\ &\quad - \int |H_t|^2 \operatorname{div} u dx - \int H \operatorname{div} u_t \cdot H_t dx. \end{aligned} \quad (4.20)$$

The estimates of the terms on the right-hand side of (4.20) are as follows:

$$\begin{aligned} - \int u_t \cdot \nabla H \cdot H_t dx &= \int \nabla H \cdot H_t \operatorname{div} u_t dx + \int u_t \cdot H \cdot \nabla H_t dx \\ &\leq C(\|\nabla u_t\|_2^2 + \|u_t\|_2^2 + \|H_t\|_2^2) + \frac{\nu}{2} \|\nabla H_t\|_2^2 \\ &\leq C(\|u_t\|_{H^1}^2 + \|H_t\|_2^2) + \frac{\nu}{2} \|\nabla H_t\|_2^2, \end{aligned}$$

$$- \int u \cdot \nabla H_t \cdot H_t dx = \frac{1}{2} \int |H_t|^2 \operatorname{div} u dx \leq C \|\nabla u\|_\infty \|H_t\|_2^2,$$

$$\int H_t \cdot \nabla u \cdot H_t dx \leq C \|\nabla u\|_\infty \|H_t\|_2^2,$$

$$\int H \cdot \nabla u_t \cdot H_t dx \leq C \|H\|_\infty \|\nabla u_t\|_2 \|H_t\|_2 \leq C(\|\nabla u_t\|_2^2 + \|H_t\|_2^2),$$

$$- \int |H_t|^2 \operatorname{div} u dx \leq C \|\nabla u\|_\infty \|H_t\|_2^2,$$

$$- \int H \operatorname{div} u_t \cdot H_t dx \leq C(\|\nabla u_t\|_2^2 + \|H_t\|_2^2).$$

Thus from (4.20), it is not difficult to prove that

$$\frac{d}{dt} \|H_t\|_2^2 + \|\nabla H_t\|_2^2 \lesssim \|u_t\|_{H^1}^2 + \|H_t\|_2^2 (1 + \|\nabla u\|_\infty),$$

which by the Gronwall's inequality gives that

$$\sup_{t \in [0, T]} \|H_t\|_2^2 + \int_0^T \|\nabla H_t\|_2^2 dt \lesssim e^{\int_0^T (1 + \|\nabla u\|_\infty) dt} (\|H_t^\delta(0)\|_2^2 + \int_0^T \|H_t\|_{H^1}^2 dt) \leq C.$$

By the magnetic equation of (2.4), we have

$$\begin{aligned} \|\nabla^2 H\|_2 &\leq \|H_t\|_2 + \|u \cdot \nabla H\|_2 + \|H \cdot \nabla u\|_2 + \|H \operatorname{div} u\|_2 \\ &\leq \|H_t\|_2 + \|u\|_\infty \|\nabla H\|_2 + \|H\|_\infty \|\nabla u\|_2 \leq C. \end{aligned}$$

The estimate of $\int_0^T \|\nabla^2 H\|_q^2 dt \leq C$ is fairly easy so we skip it.

It remains for us to show the estimate (4.19). For this, Multiplying the equation (4.15) by u_{tt} , and then integrating in the space variable x give that

$$\begin{aligned} & \|\sqrt{\rho}u_{tt}\|_2^2 + \frac{1}{2} \frac{d}{dt} \int \mu |\nabla u_t|^2 + (\mu + \lambda(\rho)) |\operatorname{div} u_t|^2 dx \\ &= \frac{1}{2} \int \lambda(\rho)_t |\operatorname{div} u_t|^2 dx - \int [\nabla p + \frac{1}{2} \nabla |H|_t^2 - H_t \cdot \nabla H - H \cdot \nabla H_t \\ & \quad + \rho_t u_t + \rho_t u \cdot \nabla u_t + \rho u \cdot \nabla u_t + \rho u_t \cdot \nabla u - \nabla(\lambda(\rho)_t \operatorname{div} u)] \cdot u_{tt} dx. \end{aligned} \quad (4.21)$$

Note that

$$\begin{aligned} & \int \nabla(\lambda(\rho)_t \operatorname{div} u) \cdot u_{tt} dx = - \int \lambda(\rho)_t \operatorname{div} u \operatorname{div} u_{tt} dx \\ &= - \frac{d}{dt} \int \lambda(\rho)_t \operatorname{div} u \operatorname{div} u_t dx + \int \lambda(\rho)_t |\operatorname{div} u_t|^2 + \lambda(\rho)_{tt} \operatorname{div} u \operatorname{div} u_t dx. \end{aligned}$$

Hence we write (4.21) as

$$\begin{aligned} & \|\sqrt{\rho}u_{tt}\|_2^2 + \frac{1}{2} \frac{d}{dt} \int \mu |\nabla u_t|^2 + (\mu + \lambda(\rho)) |\operatorname{div} u_t|^2 + \lambda(\rho)_t \operatorname{div} u \operatorname{div} u_t dx \\ &= \frac{3}{2} \int \lambda(\rho)_t |\operatorname{div} u_t|^2 dx - \int [\nabla p + \frac{1}{2} \nabla |H|_t^2 - H_t \cdot \nabla H - H \cdot \nabla H_t \\ & \quad + \rho_t u_t + \rho_t u \cdot \nabla u_t + \rho u \cdot \nabla u_t + \rho u_t \cdot \nabla u] \cdot u_{tt} dx + \int \lambda(\rho)_{tt} \operatorname{div} u \operatorname{div} u_t dx. \end{aligned} \quad (4.22)$$

Observe that $\lambda(\rho)$ satisfies the transport equation $\lambda(\rho)_t = -u \cdot \nabla \lambda(\rho) - \rho \lambda'(\rho) \operatorname{div} u$, then it holds that

$$\begin{aligned} & \frac{3}{2} \int \lambda(\rho)_t |\operatorname{div} u_t|^2 dx = - \frac{3}{2} \int u \cdot \nabla \lambda(\rho) |\operatorname{div} u_t|^2 dx - \frac{3}{2} \int \rho \lambda'(\rho) |\operatorname{div} u_t|^2 dx \\ &= 3 \int \lambda(\rho) \operatorname{div} u_t u \cdot \nabla \operatorname{div} u_t dx + \frac{3}{2} \int (\lambda(\rho) - \rho \lambda'(\rho)) \operatorname{div} u |\operatorname{div} u_t|^2 dx \\ &\leq C \|\lambda(\rho) u\|_\infty \|\operatorname{div} u_t\|_2 \|\nabla \operatorname{div} u_t\|_2 + C \|\lambda(\rho) - \rho \lambda'(\rho)\|_\infty \|\nabla u\|_\infty \|\lambda(\rho) u\|_\infty \|\operatorname{div} u_t\|_2^2 \\ &\leq C \|\operatorname{div} u_t\|_2 \|\nabla \operatorname{div} u_t\|_2 + C \|\nabla u\|_\infty \|\operatorname{div} u_t\|_2^2. \end{aligned} \quad (4.23)$$

From (4.15), it follows that

$$\mathcal{L}_\rho u_t = \rho u_{tt} + \rho_t u_t + (\rho u \cdot \nabla u)_t + \nabla p(\rho)_t + \nabla \left(\frac{1}{2} |H|_t^2 \right) - (H \cdot \nabla H)_t - \nabla(\lambda(\rho)_t \operatorname{div} u).$$

Then the standard elliptic estimates show that

$$\begin{aligned} \|\nabla^2 u_t\|_2 &\leq C [\|\sqrt{\rho}\|_\infty \|\sqrt{\rho} u_{tt}\|_2 + \|\rho_t\|_4 \|u_t\|_4 + \|\rho_t\|_4 \|\nabla u\|_4 \|u\|_\infty \\ & \quad + \|\nabla u\|_4 \|u_t\|_4 \|\rho\|_\infty + \|u\|_\infty \|\rho\|_\infty \|\nabla u_t\|_2 + \|\nabla p(\rho)_t\|_2 \\ & \quad + \|\nabla H\|_4 \|H_t\|_4 + \|H\|_\infty \|\nabla H_t\|_2 + \|\nabla \lambda(\rho)_t\|_2 \|\nabla u\|_\infty \\ & \quad + \|\lambda(\rho)_t\|_4 \|\nabla^2 u\|_4] \\ &\leq C [\|\sqrt{\rho} u_{tt}\|_2 + \|u_t\|_4 + \|H_t\|_4 + \|\nabla u_t\|_2 + \|\nabla H_t\|_2 \\ & \quad + \|\nabla^2 u\|_4 + \|\nabla u\|_\infty + 1] \\ &\leq C [\|\sqrt{\rho} u_{tt}\|_2 + \|u_t\|_4 + \|H_t\|_4 + \|\nabla u_t\|_2 + \|\nabla H_t\|_2 \\ & \quad + \|\nabla^3 u\|_2 + \|\nabla u\|_\infty + 1]. \end{aligned} \quad (4.24)$$

Plugging (4.24) into (4.23) leads to

$$\begin{aligned} \frac{3}{2} \int \lambda(\rho)_t |\operatorname{div} u_t|^2 dx &\leq \frac{1}{16} \|\sqrt{\rho} u_{tt}\|_2^2 + C(\|u_t\|_4^2 + \|H_t\|_4^2 + \|\nabla H_t\|_2^2 \\ &\quad + \|\nabla^3 u\|_2^2 + \|\nabla u\|_\infty^2) + C(\|\nabla u\|_\infty + 1) \|\nabla u_t\|_2^2. \end{aligned} \quad (4.25)$$

On the other hand, it holds that

$$\begin{aligned} - \int \nabla p(\rho)_t \cdot u_{tt} dx &= \int p(\rho)_t \operatorname{div} u_{tt} dx \\ &= \frac{d}{dt} \int p(\rho)_t \operatorname{div} u_t dx - \int p(\rho)_{tt} \operatorname{div} u_t dx \\ &\leq \frac{d}{dt} \int p(\rho)_t \operatorname{div} u_t dx + \|p(\rho)_{tt}\|_2^2 + \|\operatorname{div} u_t\|_2^2, \end{aligned} \quad (4.26)$$

$$\begin{aligned} \int \rho_t u_t \cdot u_{tt} dx &= - \int \rho_t \left(\frac{|u_t|^2}{2} \right)_t dx = - \frac{d}{dt} \int \rho_t \frac{|u_t|^2}{2} dx + \int \rho_{tt} \frac{|u_t|^2}{2} dx \\ &= - \frac{d}{dt} \int \rho_t \frac{|u_t|^2}{2} dx - \int \operatorname{div}(\rho u)_t \frac{|u_t|^2}{2} dx \\ &= - \frac{d}{dt} \int \rho_t \frac{|u_t|^2}{2} dx - \int (\rho u)_t \cdot \nabla u_t \cdot u_t dx \\ &\leq - \frac{d}{dt} \int \rho_t \frac{|u_t|^2}{2} dx + \|\sqrt{\rho}\|_\infty \|\sqrt{\rho} u_t\|_2 \|u_t\|_4 \|\nabla u_t\|_4 \\ &\quad + \|u\|_\infty \|\rho_t\|_4 \|u_t\|_4 \|\nabla u_t\|_2 \\ &\leq - \frac{d}{dt} \int \rho_t \frac{|u_t|^2}{2} dx + C(\|u_t\|_4 \|\nabla u_t\|_4 + \|u_t\|_4 \|\nabla u_t\|_2) \\ &\leq - \frac{d}{dt} \int \rho_t \frac{|u_t|^2}{2} dx + C(\|u_t\|_4 \|\nabla^2 u_t\|_2 + \|u_t\|_4 \|\nabla u_t\|_2) \\ &\leq - \frac{d}{dt} \int \rho_t \frac{|u_t|^2}{2} dx + C\|u_t\|_4 [\|\sqrt{\rho} u_{tt}\|_2 + \|u_t\|_4 + \|H_t\|_4 \\ &\quad + \|\nabla u_t\|_2 + \|\nabla H_t\|_2 + \|\nabla^3 u\|_2 + \|\nabla u\|_\infty + 1] \\ &\leq - \frac{d}{dt} \int \rho_t \frac{|u_t|^2}{2} dx + \frac{1}{16} \|\sqrt{\rho} u_{tt}\|_2^2 + C[\|u_t\|_4^2 + \|\nabla u_t\|_2^2 \\ &\quad + \|\nabla H_t\|_2^2 + \|\nabla^3 u\|_2^2 + \|\nabla u\|_\infty^2 + 1], \end{aligned} \quad (4.27)$$

$$\begin{aligned} - \int \nabla \left(\frac{1}{2} |H_t|^2 \right) \cdot u_{tt} dx &= - \int \nabla H \cdot H_t \cdot u_{tt} dx - \int H \cdot \nabla H_t \cdot u_{tt} dx \\ &\leq \|\sqrt{\rho} u_{tt}\|_2 \left\| \frac{1}{\sqrt{\rho}} \right\|_\infty \|\nabla H\|_4 \|H_t\|_4 + \|H\|_\infty \|\nabla H_t\|_2 \|\sqrt{\rho} u_{tt}\|_2 \left\| \frac{1}{\sqrt{\rho}} \right\|_\infty \\ &\leq \frac{1}{16} \|\sqrt{\rho} u_{tt}\|_2^2 + C\|\nabla H_t\|_2^2, \end{aligned} \quad (4.28)$$

$$\begin{aligned} \int H_t \cdot \nabla H \cdot u_{tt} dx &\leq \|\sqrt{\rho} u_{tt}\|_2 \left\| \frac{1}{\sqrt{\rho}} \right\|_\infty \|\nabla H\|_4 \|H_t\|_4 \leq \frac{1}{16} \|\sqrt{\rho} u_{tt}\|_2^2 + C\|\nabla H_t\|_2^2, \end{aligned} \quad (4.29)$$

$$\int H \cdot \nabla H_t \cdot u_{tt} dx \leq \|\sqrt{\rho} u_{tt}\|_2 \|\frac{1}{\sqrt{\rho}}\|_\infty \|H\|_\infty \|\nabla H_t\|_2 \leq \frac{1}{16} \|\sqrt{\rho} u_{tt}\|_2^2 + C \|\nabla H_t\|_2^2, \quad (4.30)$$

$$\begin{aligned} & - \int \rho_t u \cdot \nabla u \cdot u_{tt} dx \\ &= - \frac{d}{dt} \int \rho_t u \cdot \nabla u \cdot u_t dx + \int \rho_{tt} u \cdot \nabla u \cdot u_t dx + \int \rho_t u_t \cdot \nabla u \cdot u_t dx \\ &+ \int \rho_t u \cdot \nabla u_t \cdot u_t dx \\ &\leq - \frac{d}{dt} \int \rho_t u \cdot \nabla u \cdot u_t dx + \|\rho_{tt}\|_2 \|u\|_\infty \|\nabla u\|_4 \|u_t\|_4 + \|\rho_t\|_4 \|\nabla u\|_4 \|u_t\|_4^2 \\ &+ \|\rho_t\|_4 \|u\|_\infty \|\nabla u_t\|_2 \|u_t\|_4 \\ &\leq - \frac{d}{dt} \int \rho_t u \cdot \nabla u \cdot u_t dx + C(\|\rho_{tt}\|_2 \|u_t\|_4 + \|u_t\|_4^2 + \|\nabla u_t\|_2 \|u_t\|_4) \\ &\leq - \frac{d}{dt} \int \rho_t u \cdot \nabla u \cdot u_t dx + C(\|\rho_{tt}\|_2^2 + \|u_t\|_4^2 + \|\nabla u_t\|_2^2), \end{aligned} \quad (4.31)$$

$$- \int \rho u \cdot \nabla u_t \cdot u_{tt} dx \leq \|\sqrt{\rho} u_{tt}\|_2 \|\sqrt{\rho} u\|_\infty \|\nabla u_t\|_2 \leq \frac{1}{16} \|\sqrt{\rho} u_{tt}\|_2^2 + C \|\nabla u_t\|_2^2, \quad (4.32)$$

$$- \int \rho u_t \cdot \nabla u \cdot u_{tt} dx \leq \|\sqrt{\rho} u_{tt}\|_2 \|\sqrt{\rho}\|_\infty \|\nabla u\|_4 \|u_t\|_4 \leq \frac{1}{16} \|\sqrt{\rho} u_{tt}\|_2^2 + C \|u_t\|_4^2 \quad (4.33)$$

and

$$\int \lambda(\rho)_{tt} \operatorname{div} u \operatorname{div} u_t dx \leq \|\lambda(\rho)_{tt}\|_2 \|\nabla u\|_\infty \|\nabla u_t\|_2 \leq \frac{1}{2} (\|\lambda(\rho)_{tt}\|_2^2 + \|\nabla u\|_\infty^2 \|\nabla u_t\|_2^2). \quad (4.34)$$

Collecting all the above estimates and plugging them into (4.22) yields that

$$\begin{aligned} & \frac{1}{2} \|\sqrt{\rho} u_{tt}\|_2^2 + \frac{d}{dt} \int \mu |\nabla u_t|^2 + (\mu + \lambda(\rho)) |\operatorname{div} u_t|^2 + \lambda(\rho)_t \operatorname{div} u \operatorname{div} u_t \\ & - p(\rho)_t \operatorname{div} u_t + \rho_t \frac{|u_t|^2}{2} + \rho_t u \cdot \nabla u \cdot u_t dx \\ & \leq C[\|(\rho_{tt}, p(\rho)_{tt}, \lambda(\rho)_{tt})\|_2^2 + \|u_t\|_4^2 + \|H_t\|_4^2 + \|\nabla H_t\|_2^2 + \|\nabla^3 u\|_2^2 \\ & + (\|\nabla u\|_\infty^2 + 1)(\|\nabla u_t\|_2^2 + 1)]. \end{aligned} \quad (4.35)$$

Notice that

$$\begin{aligned} | \int \lambda(\rho)_t \operatorname{div} u \operatorname{div} u_t dx | & \leq \|\lambda(\rho)_t\|_4 \|\operatorname{div} u\|_4 \|\nabla u_t\|_4 \\ & \leq \frac{\mu}{8} \|\nabla u_t\|_2^2 + C \|\lambda(\rho)_t\|_4^2 \|\nabla u\|_4^2 \\ & \leq \frac{\mu}{8} \|\nabla u_t\|_2^2 + C, \end{aligned}$$

$$| - \int p(\rho)_t \operatorname{div} u_t dx | \leq \frac{\mu}{8} \|\nabla u_t\|_2^2 + C \|p(\rho)_t\|_2^2 \leq \frac{\mu}{8} \|\nabla u_t\|_2^2 + C,$$

$$\begin{aligned}
|\int \rho_t \frac{|u_t|^2}{2} dx| &= |\int \operatorname{div}(\rho u) \frac{|u_t|^2}{2} dx| = |\int \rho u \cdot \nabla u_t \cdot u_t dx| \\
&\leq \|\sqrt{\rho} u_t\|_2 \|\sqrt{\rho} u\|_\infty \|\nabla u_t\|_2 \leq \frac{\mu}{8} \|\nabla u_t\|_2^2 + C, \\
|\int \rho_t u \cdot \nabla u \cdot u_t dx| &= |\int \operatorname{div}(\rho u) (u \cdot \nabla u \cdot u_t) dx| = |\int \rho u \cdot \nabla (u \cdot \nabla u \cdot u_t) dx| \\
&\leq \|\sqrt{\rho} u_t\|_2 \|\sqrt{\rho} u\|_\infty (\|\nabla u\|_4^2 + \|u\|_\infty \|\nabla^2 u\|_2) + \|\rho |u|^2\|_\infty \|\nabla u_t\|_2 \|\nabla u\|_2 \\
&\leq \frac{\mu}{8} \|\nabla u_t\|_2^2 + C.
\end{aligned}$$

Thus, for some positive constant C, C_1 , we have

$$C_1(\|\nabla u_t\|_2^2 - 1) \leq G(t) \leq C(\|\nabla u_t\|_2^2 + 1), \quad (4.36)$$

with $G(t) = \int \mu |\nabla u_t|^2 + (\mu + \lambda(\rho)) |\operatorname{div} u_t|^2 + \lambda(\rho)_t \operatorname{div} u \operatorname{div} u_t - p(\rho)_t \operatorname{div} u_t + \rho_t \frac{|u_t|^2}{2} + \rho_t u \cdot \nabla u \cdot u_t dx$. From (4.35), it holds that

$$\begin{aligned}
\frac{1}{2} \|\sqrt{\rho} u_{tt}\|_2^2 + \frac{d}{dt} G(t) &\leq C[\|(\rho_{tt}, p(\rho)_{tt}, \lambda(\rho)_{tt})\|_2^2 + \|u_t\|_4^2 + \|H_t\|_4^2 \\
&\quad + \|\nabla H_t\|_2^2 + \|\nabla^3 u\|_2^2 + (\|\nabla u\|_\infty^2 + 1)(G(t) + 1)].
\end{aligned} \quad (4.37)$$

Multiplying the inequality (4.37) by t and integrating the resulted equation in t over $[\tau, t_1]$ with $\tau, t_1 \in [0, T]$ give that

$$\begin{aligned}
&\int_\tau^{t_1} t \|\sqrt{\rho} u_{tt}\|_2^2 dt + t_1 G(t_1) \\
&\leq C\tau G(\tau) + C \int_\tau^{t_1} [\|(\rho_{tt}, p(\rho)_{tt}, \lambda(\rho)_{tt})\|_2^2 + \|u_t\|_4^2 + \|H_t\|_4^2 + \|\nabla H_t\|_2^2 \\
&\quad + \|\nabla^3 u\|_2^2 + G(t)] dt + C \int_\tau^{t_1} [(\|\nabla u\|_\infty^2 + 1)(tG(t) + 1)] dt.
\end{aligned} \quad (4.38)$$

It follows from Lemma 4.3 and (4.36) that $G(t) \in L^1(0, T)$. Hence, due to [14], there exists a subsequence τ_k such that

$$\tau_k \rightarrow 0, \quad \tau_k G(\tau_k) \rightarrow 0, \quad \text{as } k \rightarrow +\infty.$$

Take $\tau = \tau_k$ in (4.38), then $k \rightarrow +\infty$ and applying the Gronwall's inequality, one shows that

$$\sup_{t \in [0, T]} [t \|\nabla u_t\|_2^2] + \int_0^T t \|\sqrt{\rho} u_{tt}\|_2^2 dt \leq C. \quad (4.39)$$

Note that (4.24) gives that

$$\sup_{t \in [0, T]} [t \|(\rho_{tt}, p(\rho)_{tt}, \lambda(\rho)_{tt})\|_2^2] + \int_0^T t \|\nabla^2 u_t\|_2^2 dt \leq C. \quad (4.40)$$

Since $u_t = (L, B)^t - u \cdot \nabla u + \frac{1}{\rho} H \cdot \nabla H$, we prove that

$$\nabla(L, B)^t = \nabla u_t + \nabla(u \cdot \nabla u) - \nabla\left(\frac{1}{\rho} H \cdot \nabla H\right).$$

Consequently, it holds that

$$\sup_{t \in [0, T]} [t \|\nabla(L, B)^t\|_2^2] \leq C, \quad (4.41)$$

which together with (3.58) implies that

$$\sup_{t \in [0, T]} [t \|(L, B)^t\|_2^2] \leq C \sup_{t \in [0, T]} [t \|\nabla(L, B)^t\|_2^2] \leq C. \quad (4.42)$$

Thus it holds that

$$\sup_{t \in [0, T]} [t \|u_t\|_2^2] \leq C \sup_{t \in [0, T]} [t \|(L, B)^t\|_2^2 + t \|u \cdot \nabla u\|_2^2 + t \|\frac{1}{\rho} H \cdot \nabla H\|_2^2] \leq C. \quad (4.43)$$

Clearly,

$$\sup_{t \in [0, T]} [t \|u_t\|_{H^1}^2] + \int_0^T t \|u_t\|_{H^2}^2 dt \leq C. \quad (4.44)$$

Applying $\partial_{x_i x_j}$, $i, j = 1, 2$, to the continuity equation in (1.1) leads to

$$\begin{aligned} & (\rho_{x_i x_j})_t + u \cdot \nabla(\rho_{x_i x_j}) + u_{x_i x_j} \cdot \nabla \rho + u_{x_i} \cdot \nabla \rho_{x_j} + \rho_{x_i x_j} \operatorname{div} u + \rho_{x_i} (\operatorname{div} u)_{x_j} \\ & + \rho_{x_j} (\operatorname{div} u)_{x_i} + \rho (\operatorname{div} u)_{x_i x_j} = 0. \end{aligned}$$

Multiplying the above equation by $q |\nabla^2 \rho|^{q-2} \rho_{x_i x_j}$ with $q > 2$ given in Theorem (1.1) and summing over $i, j = 1, 2$, and then integrating the resulting equation with respect to x over \mathbb{T}^2 shows that

$$\begin{aligned} \frac{d}{dt} \|\nabla^2 \rho\|_q^q & \leq (q-1) \|\nabla u\|_\infty \|\nabla^2 \rho\|_q^q + Cq \|\nabla^2 \rho\|_q^{q-1} [\|\nabla \rho\|_{2q} \|\nabla^2 u\|_{2q} \\ & + \|\nabla u\|_\infty \|\nabla \rho\|_{2q} + \|\rho\|_\infty \|\nabla^3 u\|_q], \end{aligned}$$

which leads to

$$\begin{aligned} \frac{d}{dt} \|\nabla^2 \rho\|_q & \leq C [\|\nabla u\|_\infty \|\nabla^2 \rho\|_q + \|\nabla \rho\|_{2q} \|\nabla^2 u\|_{2q} + \|\rho\|_\infty \|\nabla^3 u\|_q] \\ & \leq C [\|\nabla u\|_\infty \|\nabla^2 \rho\|_q + \|\nabla^2 u\|_{W^{1,q}}], \end{aligned} \quad (4.45)$$

with $q > 2$. In a similar fashion as (4.45), one can deduce that

$$\frac{d}{dt} \|\nabla^2 p\|_q \leq C [\|\nabla u\|_\infty \|\nabla^2 p\|_q + \|\nabla^2 u\|_{W^{1,q}}]. \quad (4.46)$$

Apply ∂_{x_i} with $i = 1, 2$ to the elliptic system

$$\mathcal{L}_\rho u = \rho u_t + \rho u \cdot \nabla u - H \cdot \nabla H + \nabla p(\rho) + \nabla \left(\frac{1}{2} |H|^2 \right)$$

to get

$$\begin{aligned} \mathcal{L}_\rho u_{x_i} & = -\nabla(\lambda(\rho)_{x_i} \operatorname{div} u) + \rho_{x_i} u_t + \rho u_{x_i t} + \rho_{x_i} u \cdot \nabla u + \rho u_{x_i} \cdot \nabla u + \rho u \cdot \nabla u_{x_i} \\ & + \nabla p(\rho)_{x_i} + \nabla \left(\frac{1}{2} |H|_{x_i}^2 \right) - H_{x_i} \cdot \nabla H - H \cdot \nabla H_{x_i} := \Psi. \end{aligned}$$

Then the standard elliptic estimates give that

$$\begin{aligned} \|\nabla u\|_{W^{2,q}} & \leq C [\|\nabla u\|_q + \|\Psi\|_q] \\ & \leq C [1 + (\|\nabla u\|_\infty + 1) (\|\nabla^2 \rho, \nabla^2 p\|_q + \|\nabla^2 u\|_{2q} + \|u_t\|_{W^{1,q}} + \|\nabla^2 H\|_q)], \end{aligned}$$

which incorporating (4.45) and (4.46) implies that

$$\begin{aligned} \frac{d}{dt} \|(\nabla^2 \rho, \nabla^2 p)\|_q &\leq C[1 + (\|\nabla u\|_\infty + 1)] \|(\nabla^2 \rho, \nabla^2 p)\|_q + \|u\|_{H^3} + \|u_t\|_{H^1} \\ &\quad + \|\nabla u_t\|_q + \|\nabla^2 H\|_q. \end{aligned} \quad (4.47)$$

From Lemma 2.2, we have

$$\int_0^T \|\nabla u_t\|_q dt \leq C \int_0^T \|\nabla^2 u_t\|_2 dt \leq C \sup_{t \in [0, T]} [\sqrt{t} \|\nabla u_t\|_2] \int_0^T t^{-\frac{1}{2}} dt \leq C.$$

Therefore, from (4.47) and the Gronwall's inequality, it holds that

$$\begin{aligned} \|(\nabla^2 \rho, \nabla^2 p)\|_q &\leq \|(\nabla^2 \rho_0, \nabla^2 p(\rho_0))\|_q + C \int_0^t (1 + \|u\|_{H^3} + \|u_t\|_{H^1} \\ &\quad + \|\nabla u_t\|_q) ds \times e^{C \int_0^t (\|\nabla u\|_\infty + 1) ds} \\ &\leq C, \end{aligned} \quad (4.48)$$

which then leads to

$$\sup_{t \in [0, T]} \|(\rho, p(\rho))\|_{W^{2, q}} \leq C.$$

Hence Lemma 4.6 is now proved. \square

5. PROOF OF THEOREM 1.1

In this section we complete the proof of Theorem 1.1.

Proof of Theorem 1.1. From the uniform bounds in Lemmas 3.1-3.7 and Lemma 4.1-4.6, we can show the solution sequence (ρ^n, u^n, H^n) converges to a limit (ρ, u, H) satisfying the same bounds as (ρ^n, u^n, H^n) when $n \rightarrow \infty$ and the limit (ρ, u, H) is the unique solution to the original problem (1.1)-(1.3). We omit the details for brevity. Now, we will prove that (ρ, u, H) satisfy the bounds in Theorem 1.1 and (ρ, u, H) is in fact a classical solution to (1.1).

Note that

$$\begin{aligned} (u, H) &\in L^2(0, T; H^3(\mathbb{T}^2)) \times L^2(0, T; H^3(\mathbb{T}^2)), \\ (u_t, H_t) &\in L^2(0, T; H^1(\mathbb{T}^2)) \times L^2(0, T; H^1(\mathbb{T}^2)), \end{aligned}$$

then by Sobolev embedding theorem, one obtains

$$\begin{aligned} u &\in C([0, T]; H^2(\mathbb{T}^2)) \hookrightarrow C([0, T] \times \mathbb{T}^2), \\ H &\in C([0, T]; H^2(\mathbb{T}^2)) \hookrightarrow C([0, T] \times \mathbb{T}^2). \end{aligned}$$

After a similar argument, from $(\rho, p(\rho)) \in L^\infty([0, T]; W^{2, q}(\mathbb{T}^2))$ and $(\rho_t, p(\rho)_t) \in L^\infty([0, T]; H^1(\mathbb{T}^2))$, clearly it holds that

$$(\rho, p(\rho)) \in C([0, T]; W^{1, q}(\mathbb{T}^2)) \cap C([0, T]; W_{weak}^{2, q}(\mathbb{T}^2)).$$

Thus, from Lemma 4.6, clearly $(\rho, p(\rho)) \in C([0, T]; W^{2, q}(\mathbb{T}^2))$. Theorem 1.1 is proved. \square

Acknowledgements The authors would like to thank Prof. Y. Wang for his value discussion.

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